Embedded metasymplectic spaces of type $\mathsf{F}_{4,4}$

Abstract

We determine the generating and embedding rank of the metasymplectic spaces whose symplecta are either symplectic polar spaces in characteristic distinct from 2, or Hermitian polar spaces (including the quaternion case), and provide a characterisation of the associated projective varieties in the context of the Freudenthal-Tits magic square.

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1 Introduction

In a recent manuscript [12] the exact embedding rank and exact generating rank of the Lie incidence geometry $\mathsf{F}_{4,4}(\mathbb{K})$ related to the split building of type F_4 over a field $\mathbb{K}$ of characteristic not 2 was determined by considering as points the vertices of type 4. In this paper we primarily extend that result to the nonsplit (separable) case. More exactly, using the notation introduced in Section 2.4 below, we prove:

Theorem A. *The generating and embedding ranks of* $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ *are both equal to*

$$\begin{cases} 26, & \text{if } \mathbb{A} = \mathbb{K} \text{ and } \mathsf{char}\,\mathbb{K} \neq 2, \\ 27, & \text{if } \mathbb{A} \text{ is a separable quadratic extension of } \mathbb{K}, \\ 28, & \text{if } \mathbb{A} \text{ is a quaternion algebra over } \mathbb{K}. \end{cases}$$

The result is quite neat: it turns out that such geometries have embedding and generating ranks both equal to either 26, 27 or 28. The same series of numbers appears in Wilson's paper [37], where the author notes that the split complex Lie groups of type F_4, E_6 and E_7 can be constructed with real, complex and quaternion matrices of dimension 26, 27 and 28, respectively. In fact, this is not a coincidence: the construction can indeed be linked to the embeddings described in the present paper by splitting the division algebras: the Lie incidence geometry $\mathsf{F}_{4,4}(\mathbb{R}, \mathbb{C})$ is so to speak a nonsplit form of the Lie incidence geometry $\mathsf{E}_{6,1}(\mathbb{C})$, whereas the same is true for $\mathsf{F}_{4,4}(\mathbb{R}, \mathbb{H})$ and $\mathsf{E}_{7,6}(\mathbb{L})$. Here, $\mathbb{R}, \mathbb{C}, \mathbb{H}$ are the real, complex and quaternion division rings. However, our results hold for arbitrary fields.

The embeddings themselves are perhaps not new (although we could not find a mention in the literature of the quaternion case); they arise from Galois descent in a rather standard way. However, the determination of the embedding and generating ranks was not even known when the underlying field is finite! Along the way, we complete the determination of all embedding and generating ranks of some embeddable dual polar spaces by considering the ones of type $\mathsf{B}_{n,n}$ over a field $\mathbb{K}$ with corresponding anisotropic form given by a separable quadratic field extension, or the norm of a quaternion division algebra. For the case of a field extension, this was already known, see [9]. Our proof is only slightly different. However, it holds across all possible types, including the quaternion case.

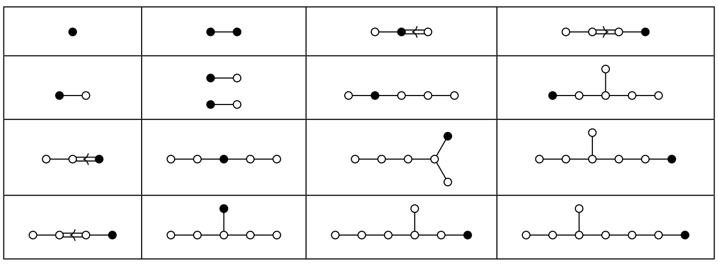
In the second part of the manuscript, we characterise these embeddings and their so-called *admissible quotients* by axioms in the spirit of [27], which grew out of a modest axiom system for finite Veronese surfaces in odd characteristic in [21], until it was turned into a powerful recognition result for embedded Lie incidence geometries. If we call a point set satisfying our axioms an *abstract metasymplectic variety* (we refer to Section 5 for the details), then more precisely we will prove the following theorem.

Theorem B. *Every abstract metasymplectic variety is an admissible quotient of the absolute universal embedding of* $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$*, with* $\mathbb{A}$ *an associative quadratic division algebra over* $\mathbb{K}$*.*

For the definition and a discussion about the admissible quotients, see Section 6. This result should be viewed in the context of the Freudenthal-Tits magic square. The latter has a number of different appearances, see [33]. Here, we are mainly interested in the Complexified Geometric Magic Square, see Section 6.7 of [33]. With the notation and conventions that we shall introduce in Section 2.4, it is the following table of types of Lie incidence geometries:

$\mathsf{A}_{1,1}$	$\mathsf{A}_{2,\{1,2\}}$	$\mathsf{C}_{3,2}$	$\mathsf{F}_{4,4}$
$\mathsf{A}_{2,1}$	$\mathsf{A}_{2,1}\times\mathsf{A}_{2,1}$	$\mathsf{A}_{5,2}$	$\mathsf{E}_{6,1}$
$\mathsf{C}_{3,3}$	$\mathsf{A}_{5,3}$	$\mathsf{D}_{6,6}$	$\mathsf{E}_{7,7}$
$\mathsf{F}_{4,1}$	$\mathsf{E}_{6,2}$	$\mathsf{E}_{7,1}$	$\mathsf{E}_{8,8}$

or, pictorially,



Each of the geometries appearing in the square has a natural representation (or embedding) in some projective space, and so we may conceive the above square as a table of projective varieties. For instance, the variety of type $\mathsf{A}_{1,1}$ is simply a conic, and that of type $\mathsf{A}_{2,1}$ is a classical Veronese variety in projective 5-space. Such a variety is, in the complex case, a Severi variety and is even characterised as such (in the given dimension 5). In 1984, Mazzocca & Melone [21] characterised the Veronese varieties over finite fields of odd order by some properties that are immediate consequences of being a Severi variety, but can be stated in terms independent from algebraic geometry. For instance one of the axioms is that every pair of points is contained in a (unique) plane conic. The same axioms were used in [26, 27], suitably adopted to higher dimensions, to characterise all of the varieties in the second row of the Magic Square above, hence including the famous $\mathsf{E}_{6,1}$-variety (also sometimes referred to as the *Cartan variety*). For instance the above mentioned axiom extends to "every pair of points is contained in a split quadric". This work was then continued in [15] to characterise the varieties of the third row with axioms in the very same spirit, once again showcasing that these varieties form a unit. The last row consists of the Weyl embeddings of the long root subgroup geometries of exceptional type. One of the difficulties arising here is that the geometries contain so-called *special pairs*, that is, pairs of points at distance 2 not contained in a common quadric. This phenomenon, however, also occurs in the first row. The varieties most closely resembling those on the fourth row are our target geometries appearing in the North-East corner of the square, the ones of type $\mathsf{F}_{4,4}$. So it seems like a useful preparatory challenge to characterise those in the same sprit as the second and third row. In addition to the fact that special pairs of points are turning up, two supplementary difficulties arise. First of all, the geometries are not determined by a field only; each geometry is determined by a pair $(\mathbb{K}, \mathbb{A})$, where $\mathbb{K}$ is a field and $\mathbb{A}$ an alternative quadratic division algebra over $\mathbb{K}$. The non-associative case does not admit a variety, and so we may assume $\mathbb{A}$ is associative. Secondly, a pair of points at distance 2 that is not special is now contained in a Hermitian or symplectic variety, and not in a quadric. The case of symplectic varieties poses some serious problems in the proofs. Indeed, the usual requirement expressing the smoothness of the varieties consists in asking that the ambient spaces of the quadrics intersect in points of the variety. Now in the case of symplectic varieties, this axiom is void. One of the consequences is that these subgeometries need not be convex anymore. Despite the fact that it is hard to live in a nonconvex world, we nevertheless work our way around this, partly helped by the necessary additional requirement that deals with the special pairs. This forms a substantial part of the arguments. Further motivation and discussion of the axioms will be done when these are stated.

We also provide a characterisation of the embedded dual polar spaces that turn up as a residue in the metasymplectic spaces we consider. Also here, the same difficulties arise. Hence, if we call a point set satisfying our corresponding axioms an *abstract dual polar variety* (we refer to Section 4 for the details), then we will also prove the following theorem.

Theorem C. *Every abstract dual polar variety is projectively equivalent to the absolute universal embedding of* $\mathsf{B}_{3,3}(\mathbb{K}, \mathbb{A})$.

Also here, the biggest challenge is the case where two points at distance 2 are contained in a symplectic subvariety where we have to show that these subvarieties are convex. However, admissible quotients do not turn up here! The fact that we cannot ignore the admissible quotients in Theorem B will be explained at the end of the paper, in Section 6.

The paper is organized as follows. In Section 2 we gather all the preliminaries. This mainly concerns the definiton and properties of certain Lie incidence geometries, in casu, the metasymplectic spaces we are going to embed and characterize. But we also need properties of geometries related to groups of type E_7 as these are the absolute type for an important class related to quaternion division rings. We also review the definition and properties of some dual polar spaces, which turn up as residues of the metasymplectic polar spaces under investigation In Section 3 we prove our main result concerning the embedding and generating ranks. Our method requires that we first determine those of certain dual polar spaces, and that is what we indeed do first. We exhibit embeddings and prove universality. The existence is proved via Tits indices and Galois descent. In Section 4 we discuss the axiomatic approach to embedded dual polar spaces and we prove Theorem C. We conclude in Section 5 with an axiomatic characterization of metasymplectic spaces proving Theorem B.

2 Preliminaries

We introduce the geometries central in this paper, and their representations in projective space. This has two levels. First of all, the abstract level where the axiomatization of the geometries is explained, followed by the definition of certain isomorphism classes of geometries, all of them related to spherical buildings. Lastly, properties of these geometries needed in this paper are reviewed.

2.1 Abstract point-line geometries

We fix notation and introduce all relevant terminology. We assume that the reader is familiar with the basic theory of abstract buildings, Coxeter groups and Dynkin diagrams [4] and refer to the literature (for instance [1] or [30]) for precise definitions and details. We say that a spherical building is *split* if it arises from a split algebraic group.

A *point-line geometry* is a pair $\Gamma = (X, \mathscr{L})$ with X the set of points and $\mathscr{L}$ the set of lines $\mathscr{L}$ which is a subset of the power set of X. To exclude trivial cases, we assume $|\mathscr{L}| \geq 2$. We also assume that each line has at least three points.

Points $x, y \in X$ contained in a common line are called *collinear*, denoted $x \perp y$; the set of all points collinear to x is denoted by $x^\perp$. We will always deal with situations where every point is contained in at least one line, so $x \in x^\perp$. The *collinearity graph* of Γ is the graph on X with collinearity as adjacency relation. The *distance* δ between two points $p, q \in X$ (denoted $\delta_\Gamma(p,q)$, or $\delta(p,q)$ if no confusion is possible) is the distance between p and q in the collinearity graph, where $\delta(p,q) = \infty$ if there is no such path. If $\delta := \delta(p,q)$ is finite, then a *geodesic path* or a *shortest path* between p and q is a path of length δ between them in the collinearity graph. The *diameter* of Γ (denoted $\mathsf{diam}\,\Gamma$) is the diameter of the collinearity graph. We say that Γ is *connected* if every pair of vertices is at finite distance from one another. The point-line geometry Γ is called a *partial linear space* if each pair of distinct collinear points x, y is contained in exactly one line, denoted xy.

A *subspace* of Γ is a subset S of X such that, if $x, y \in S$ are collinear and distinct, then all lines containing both x and y are contained in S. A subspace S is called *convex* if, for any pair of points $\{p, q\} \subseteq S$, every point occurring in a shortest path between p and q in the collinearity graph is contained in S; it is *singular* if $\delta(p,q) \leq 1$ for all $p, q \in S$. The intersection of all convex subspaces of Γ containing a given subset $S \subseteq X$ is called the *convex closure* of S (this is well defined since X is a convex subspace). For $S \subseteq X$, we denote by $\langle S \rangle$ the subspace generated by S, it is the intersection of all subspaces containing S (again, this is well defined since X is a subspace). If S consists of two distinct collinear points p and q contained in a unique line L, then $\langle S \rangle = L$ is sometimes briefly denoted by pq. Two singular subspaces S_1 and S_2 are called *collinear* if $S_1 \cup S_2$ is a set of pairwise collinear points, and if so, we write $\langle S_1, S_2 \rangle$ instead of $\langle S_1 \cup S_2 \rangle$. In the geometries that we will consider, that is, parapolar spaces, the subspace generated by a set of mutually collinear points is always a singular subspace.

2.2 Polar spaces

Abstractly, a (nondegenerate, thick) *polar space* $\Gamma = (X, \mathscr{L})$ is a point-line geometry satisfying the following four axioms by Buekenhout and Shult, which simplifies Tits' axiom system [30].

(PS1) Every line contains at least three points, i.e., every line is *thick*.

(PS2) No point is collinear to every other point.

(PS3) Every nested sequence of singular subspaces is finite.

(PS4) The set of points incident with a given arbitrary line L and collinear to a given arbitrary point p is either a singleton or coincides with L.

We will assume that the reader is familiar with the basic theory of polar spaces, see for instance [6]. Let us recall that every polar space, as defined above, is a partial linear space and has a unique *rank*, given by the length of the longest nested sequence of singular subspaces (including the empty set); the rank is always assumed to be finite (by (PS3)) and at least 2 since we always have a sequence $\emptyset \subseteq \{p\} \subseteq L$, for a line $L \in \mathscr{L}$ and a point $p \in L$.

Now let $\Gamma = (X, \mathscr{L})$ be a polar space of rank $r \geq 2$. It is well known that the maximal singular subspaces are projective spaces of dimension $r-1$ (and so two arbitrary points of Γ are contained in at most one line). Moreover, there is a (not necessarily finite) constant t such that every singular subspace of dimension $r-2$ is contained in exactly $t+1$ maximal singular subspaces. If $t = 1$, then we say that Γ is of *hyperbolic type*, or is a *hyperbolic* polar space. A hyperbolic polar space of rank at least 3 is isomorphic to a nondegenerate hyperbolic quadric Q in $\mathsf{PG}(2r-1, \mathbb{K})$, $\mathbb{K}$ a (commutative) field. The lines are the lines of $\mathsf{PG}(2r-1, \mathbb{K})$ entirely contained in Q. Note that a standard equation for Q is given by $X_{-1}X_1 + X_{-2}X_2 + \cdots + X_{-r}X_r = 0$.

A maximal singular subspace of a hyperbolic polar space is also called a *generator*. The family of generators of each hyperbolic polar space of rank r is the disjoint union of two systems of generators, called the *natural systems*, such that two generators intersect in a singular subspace of odd codimension in each of them if, and only if, they belong to different systems (the *codimension* of a subspace U in a projective space W is just $\dim W - \dim U$).

We will use some notions of the theory of buildings in polar spaces. For instance, two subspaces are called *opposite* if no point of their union is collinear to every point of this union; in particular two points are opposite if, and only if, they are not collinear and two maximal singular subspaces are opposite if, and only if, they are disjoint.

2.3 Parapolar spaces

Parapolar spaces are point-line geometries that are designed to model the Grassmannians of spherical buildings. They were introduced by Cooperstein [8]. A standard reference is [28]. A point-line geometry $\Gamma = (X, \mathscr{L})$ is a *parapolar space* if it satisfies the following axioms.

(PPS1) There is line L and a point p such that no point of L is collinear to p.

(PPS2) The geometry is connected.

(PPS3) Let x, y be two points at distance 2. Then either there is a unique point collinear with both, or the convex closure of $\{x, y\}$ is a polar space. Such polar spaces are called *symplecta*, or *symps* for short.

(PPS4) Each line is contained in a symplecton.

A pair $\{x, y\}$ of points with $x^\perp \cap y^\perp = \{z\}$ is called *special* and we denote $z = x \bowtie y$; we also say that *x is special to y*. The set of points special to x is denoted by $x^{\bowtie}$. A pair of points $\{x, y\}$ at distance 2 from one another and contained in a (necessarily unique) symp is called *symplectic* and we write $x \perp\!\!\!\perp y$, we also say that *x is symplectic to y*. The set of points contained in a symp together with x is denoted by $x^{\perp\!\!\!\perp}$ (note that this hence also includes $x^\perp$ by (PPS4). A parapolar space without special pairs of points is called *strong*. Due to (PPS4) and the fact that symps are convex subspaces isomorphic to polar spaces, each parapolar space is automatically a partial linear space and, by (PPS1), it is not a polar space. Note that the symps are not required to all have the same rank. A *para* is a proper convex subspace of Γ, whose points and lines form a parapolar space themselves. The set of symps of a para is a subset of the set of symps of Γ.

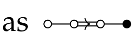
We will also make use of residues. If $\Gamma = (X, \mathscr{L})$ is a polar space of rank r, or parapolar space whose symps have rank at least r, then for a singular subspace U of dimension $d \leq r-3$, we define the residue of Γ at U, denoted by $\mathsf{Res}_\Gamma(U)$, as the point-line geometry $(X_U, \mathscr{L}_U)$, where X_U is the set of singular subspaces of dimension $d+1$ of Γ containing U, and an element of $\mathscr{L}_U$ is the set of $(d+1)$-dimensional subspaces through U contained in a singular subspace of dimension $d+2$ through U.

Embeddings of point-line geometries in each other

Consider two point-line geometries $\Gamma = (X', \mathscr{L}')$ and $\Delta = (X, \mathscr{L})$. We say that Γ is *embedded* in Δ if $X' \subseteq X$ and for each $L' \in \mathscr{L}'$, there is a line $L \in \mathscr{L}$ with L' (viewed as subset of X') contained in L (viewed as a subset of X). The embedding is called *full* if $\mathscr{L}' \subseteq \mathscr{L}$, i.e., $L' \subseteq X'$ coincides with $L \subseteq X$ in the foregoing. We will mainly apply this in the case where Δ is a projective space, and then we call the embedding a *projective embedding*. We sometimes emphasize that an embedding is not (necessarily) full by calling it *lax*.

Next, suppose additionally that $\Gamma = (X', \mathscr{L}')$ and $\Delta = (X, \mathscr{L})$ are parapolar spaces of diameter at most 3. Then we call the embedding *isometric* if it preserves the distance and being special.

2.4 Lie incidence geometries

Let Δ be a (thick) spherical building, not necessarily irreducible. Let n be its rank, let S be its type set and let $J \subseteq S$. Then we define a point-line geometry $\Gamma = (X, \mathscr{L})$ as follows. The point set X is just the set of flags of Δ of type J; each member of $\mathscr{L}$ is given by the elements F of X that complete a given flag F' of type $S \setminus \{s\}$, with $s \in J$, to a chamber, that is, $F \cup F'$ is a chamber (note that several distinct flags F' can give rise to the same line of Δ). The geometry Γ is called a *Lie incidence geometry*. For instance, if Δ has type A_n, and $J = \{1\}$ (remember we use Bourbaki labelling), then Γ is the point-line geometry of a projective space. If X_n is the Coxeter type of Δ and Γ is defined using $J \subseteq S$ as above, then we say that Γ has *type* $\mathsf{X}_{n,J}$ and we write $\mathsf{X}_{n,j}$ if $J = \{j\}$. Pictorially, we represent such geometry by the diagram X_n where we color the nodes of types in J black. For instance a geometry of type $\mathsf{F}_{4,4}$ is drawn as . We use black nodes to distinguish these diagrams from the Tits indices introduced earlier.

Most Lie incidence geometries are parapolar spaces. In particular, with the notation of Section 2.4, if $|J| = 1$, then we either have a projective space (if $\mathsf{X} = \mathsf{A}$ and J is either $\{1\}$ or $\{n\}$), a polar space (if $\mathsf{X} \in \{\mathsf{B}, \mathsf{C}, \mathsf{D}\}$ and $J = \{1\}$), or a parapolar space (in all other cases, taking into account though that $\mathsf{A}_{3,2} = \mathsf{D}_{3,1}$). The hyperbolic polar spaces correspond precisely to the Lie incidence geometries $\mathsf{D}_{n,1}$. Lie incidence geometries of type $\mathsf{B}_{n,n}$ and $\mathsf{C}_{n,n}$, $n \geq 3$, are called *dual polar spaces* and there is extensive literature about them. For basic properties of parapolar spaces such as the facts that the intersections of symps are singular subspaces, and also that the set of points collinear to a given point x and belonging to a symp $\xi \not\ni x$ is a singular subspace, we refer to Chapter 13 of [28].

If the building Δ is irreducible and its diagram X_n is simply laced, with $n \geq 3$, then the classification in [30] implies that Δ is unambiguously defined by a (skew)field $\mathbb{K}$, which is necessarily a field if X_n contains D_4 as a subdiagram. We denote Δ by $\mathsf{X}_n(\mathbb{K})$. The Lie incidence geometry

$X_{n,J}$, $J \subseteq S$, is denoted by $X_{n,J}(\mathbb{K})$. We denote the projective space $A_{n,1}(\mathbb{L})$, for a skew field $\mathbb{L}$, more traditionally by $\mathbb{P}^n(\mathbb{L})$ (in the latter notation n is allowed to be infinite).

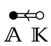
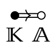
If the type of the building Δ is F_4, then by Chapter 10 of [30], Δ is unambiguously defined by a pair $(\mathbb{K}, \mathbb{A})$, where $\mathbb{K}$ is a (commutative) field and $\mathbb{A}$ is a quadratic alternative division algebra over $\mathbb{K}$. It is a custom (and explainable via the commutation relations of the root subgroups) to label the diagram in such a way that the objects corresponding to labels 1 and 2 are defined over $\mathbb{K}$, and those corresponding to the labels 3 and 4 defined over $\mathbb{A}$. We denote Δ by $F_4(\mathbb{K}, \mathbb{A})$. The geometries $F_{4,1}(\mathbb{K}, \mathbb{A})$ and $F_{4,4}(\mathbb{K}, \mathbb{A})$ are the (thick) *metasymplectic spaces*. We are especially interested in those of type $F_{4,4}$ in this paper. The maximal singular subspaces are projective planes over $\mathbb{A}$. A symplecton of $F_{4,4}(\mathbb{K}, \mathbb{A})$ is a polar space of type C_3 denoted by $C_{3,1}(\mathbb{A}, \mathbb{K})$; the symps of $F_{4,1}(\mathbb{K}, \mathbb{A})$ are denoted by $B_{3,1}(\mathbb{K}, \mathbb{A})$. The rank n analogues of these polar spaces, $n \geq 2$, are denoted by $C_{n,1}(\mathbb{A}, \mathbb{K})$ and $B_{n,1}(\mathbb{K}, \mathbb{A})$, respectively, except that $n \in \{2, 3\}$ if $\mathbb{A}$ is not associative.

2.5 Properties of some specific Lie incidence geometries

Polar and dual polar spaces of rank 3

Each polar space $C_{3,1}(\mathbb{A}, \mathbb{K})$, with $\mathbb{A}$ an associative quadratic division algebra over $\mathbb{K}$, admits a (projectively) unique full embedding in $\mathbb{P}^5(\mathbb{A})$. A *hyperbolic line* of $C_{3,1}(\mathbb{A}, \mathbb{K})$ is the set of points collinear to two given opposite lines of $C_{3,1}(\mathbb{A}, \mathbb{K})$; it coincides with the set h of points of $C_{3,1}(\mathbb{A}, \mathbb{K})$ lying on some nonsingular projective line L of $\mathbb{P}^5(\mathbb{A})$ (and each such line containing at least two points of $C_{3,1}(\mathbb{A}, \mathbb{K})$ is a hyperbolic line). The set h is a standard projective subline over $\mathbb{K}$ of the projective line L (over $\mathbb{A}$), that is, h arises from L by restricting coordinates down from $\mathbb{A}$ to $\mathbb{K}$ (with respect to an appropriate coordinatization).

Each polar space $B_{n,1}(\mathbb{K}, \mathbb{A})$ arises from a quadric in $\mathbb{P}^d(\mathbb{K})$ of Witt index n, with $d = 2n - 1 + \dim_{\mathbb{K}} \mathbb{A}$. In this case a standard equation, using coordinates in $\mathbb{K}^n \times \mathbb{A}$, where $\mathbb{A}$ is viewed as vector space over $\mathbb{K}$, is given by $x_{-n}x_n + \cdots + x_{-1}x_1 = \mathsf{N}(x)$, where $\mathsf{N}(x)$ denotes the (standard) norm of $x \in \mathbb{A}$.

Let Δ be a dual (thick) polar space of rank 3 and let Δ^* be the corresponding (thick) polar space of rank 3. The points of Δ correspond to the maximal singular subspaces of Δ^* and the lines of Δ correspond to the submaximal singular subspaces of Δ^*. We will view Δ as a strong parapolar space of diameter n. Its symps are generalised quadrangles isomorphic to $C_{2,1}(\mathbb{A}, \mathbb{K})$  if Δ is isomorphic to $B_{n,n}(\mathbb{K}, \mathbb{A})$. In particular, $C_{2,1}(\mathbb{A}, \mathbb{K})$ is the dual of $B_{2,1}(\mathbb{K}, \mathbb{A})$  .
The maximal singular subspaces of Δ are lines.

Parapolar spaces of type F_4

In the present paper, we will mainly be interested in the metasymplectic spaces $F_{4,4}(\mathbb{K}, \mathbb{A})$,  where $\mathbb{A}$ is either a quaternion algebra over $\mathbb{K}$, or a separable quadratic field extension of $\mathbb{K}$, or $\mathbb{A} = \mathbb{K}$ and has characteristic different from 2. We refer to the *separable case*, or say that $\mathbb{A}$ *is separable over* $\mathbb{K}$. The basic properties are the following, stated as facts (and we refer to [7]).

Fact 2.1. *The lines, planes and symps of* $F_{4,4}(\mathbb{K}, \mathbb{A})$ *through a given point* p*, endowed with the natural incidence relation, form a dual polar space* $\mathsf{Res}_\Gamma(p)$ *of rank* 3 *isomorphic to* $B_{3,3}(\mathbb{K}, \mathbb{A})$*, where the points* *of the corresponding polar space are the symps through p, the lines are the planes of Γ through p, and the planes are the lines of Γ through p.*

The geometry $\mathsf{Res}_\Gamma(p)$ is usually called the *point residual* at p in Γ.

Fact 2.2 (Point-point relations)**.** *Let x and y be two points of Γ. Then $\delta_\Gamma(x,y) \leq 3$ (and distance 3 occurs and corresponds to opposite points) and if $\delta_\Gamma(x,y) = 2$, then either x and y are contained in a unique symp $\xi(x,y)$, or there is a unique point $x \bowtie y$ collinear to both x and y.*

Fact 2.3 (Symp-symp relations)**.** *The intersection of two symps ξ_1 and ξ_2 is either empty, or a point, or a plane. If $\xi_1 \cap \xi_2 = \emptyset$, then either, for each point $x_1 \in \xi_1$ there is a unique point $x_2 \in \xi_2$ symplectic to x_1 (and the correspondence $x_1 \mapsto x_2$ is an isomorphism of polar spaces), or there exists a unique symp ζ intersecting ξ_1 and ξ_2 in planes which are opposite as planes of the polar space ζ. In the former case, ξ_1 is opposite ξ_2; in the latter case we say that ξ_1 and ξ_2 are special.*

Fact 2.4 (Point-symp relations)**.** *Let p be a point and Σ a symp of Γ with $p \notin \Sigma$. Then one of the following occurs:*

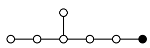
(*i*) *$p^\perp \cap \Sigma$ is line L. In this case, p and x are symplectic for all $x \in \Sigma \cap (L^\perp \setminus L)$ (and $L \subseteq \xi(p,x)$), and p and x are special for all $x \in \Sigma \setminus L^\perp$ (and $p \bowtie x \in L$). We say that p and Σ are close;*

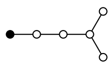
(*ii*) *$p^\perp \cap \Sigma$ is empty, but there is a unique point u of Σ symplectic to p (so $\Sigma \cap \xi(p,u) = \{u\}$). Then x and p are special for all $x \in \Sigma \cap (u^\perp \setminus \{u\})$ (and $x \bowtie p \notin \Sigma$), and x and p are opposite if $x \in \Sigma \setminus u^\perp$. We say that p and Σ are far.*

Combining Fact 2.3 and Fact 2.4(i), we obtain

Fact 2.5. *Let ξ and ζ be two symps which intersect in a unique point p. Then each line L of ξ through p is coplanar with a unique line M of ζ through p and the mapping $L \mapsto M$ is an isomorphism from $\mathsf{Res}_\xi(p)$ to $\mathsf{Res}_\zeta(p)$.*

Strong parapolar spaces of type $\mathsf{E}_{7,7}$

We will also need the Lie incidence geometry $\mathsf{E}_{7,7}(\mathbb{K})$, , since this parapolar space is the natural home of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$, for $\mathbb{A}$ a quaternion algebra over $\mathbb{K}$, as we will see below.

Let Δ be the parapolar space $\mathsf{E}_{7,7}(\mathbb{K})$. Then Δ is a strong parapolar space of diameter 3; points at distance 3 are *opposite*. A maximal singular subspace has either dimension 5 (in this case occurring as an intersection of two symps and corresponding to a type 3 element in the Dynkin diagram) or dimension 6 (type 2 in the Dynkin diagram). The 5-dimensional subspaces of a 6-space will be called *5′-spaces*. They do not correspond to a single node of the Dynkin diagram, but rather to an incident pair of nodes of type $\{1,2\}$. Each symp of Δ is isomorphic to the polar space $\mathsf{D}_{6,1}(\mathbb{K})$  (the residue of an element of type 1 in the underlying spherical building). Furthermore, the lines, planes, 3-dimensional singular subspaces and 4-dimensional subspaces correspond to types 6, 5, 4 and $\{2,3\}$ in the Dynkin diagram.

We now review the point-symp and symp-symp relations. They can be deduced by considering an appropriate model of an apartment (the "thin version") of a building of type E_7, as given in [34].

Fact 2.6 (Point-symp relations)**.** *If p is a point and ξ a symp of Δ with $p \notin \xi$, then precisely one of the following occurs.*

(*i*) *p is collinear to a unique point* $q \in \xi$. *In this case, p and x are symplectic if* $x \in \xi \cap (q^\perp \setminus \{q\})$ *and* $\delta(p,x) = 3$ *for* $x \in \xi \setminus q^\perp$. *Here, p is called close to* ξ.

(*ii*) *p is collinear to a* $5'$*-space U of* ξ. *In this case, x and p are symplectic if* $x \in \Sigma \setminus U$ *and p is called far from* ξ

This fact implies:

Corollary 2.7. *On each line L of* Δ, *there is at least one point symplectic to a given point p, unique when L contains at least one point opposite p.*

Fact 2.8 (Symp-symp relations)**.** *If* ξ *and* ξ' *are two symps of* Δ, *then precisely one of the following occurs.*

(*i*) $\xi = \xi'$;

(*ii*) $\xi \cap \xi'$ *is a* 5*-space.*

(*iii*) $\xi \cap \xi'$ *is a line L. Then points* $x \in \xi \setminus L$ *and* $x' \in \xi' \setminus L$ *are never collinear and* $\delta(x,x') = 3$ *if, and only if,* $x^\perp \cap L$ *is disjoint from* $x'^\perp \cap L$.

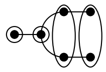
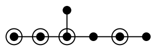
(*iv*) $\xi \cap \xi' = \emptyset$ *and there is a unique symp* ξ'' *intersecting* ξ *in a* 5*-space U and intersecting* ξ' *in a* 5*-space* U', *with U and* U' *opposite in* ξ''.

(*v*) $\xi \cap \xi' = \emptyset$ *and every point x of* ξ *is collinear to a unique point* x' *of* ξ'. *In this situation,* ξ *and* ξ' *are opposite, and the correspondence* $x \mapsto x'$ *is an isomorphism of polar spaces.*

2.6 Galois descent; Tits indices

By the classification of spherical buildings of type F_4 in Chapter 10 of [30], and the tables in [29], each building $\mathsf{F}_4(\mathbb{K},\mathbb{A})$, with $\mathbb{A}$ separable over $\mathbb{K}$, arises from a split building by so-called *Galois descent*, which we can describe here in geometric terms as follows (and our description is justified by the fact that each separable associative quadratic division algebra over $\mathbb{K}$ distinct from $\mathbb{K}$ itself splits over a suitable quadratic extension). If $\mathbb{K} = \mathbb{A}$ then $\mathsf{F}_4(\mathbb{K},\mathbb{K})$ is itself split and so there is nothing to explain. Otherwise, there is a building Δ of type E_6 or E_7, defined over $\mathbb{A}$ (if $\mathbb{A}$ is commutative, that is, if $\mathbb{A}$ is itself a quadratic extension of $\mathbb{K}$) or over a subfield of $\mathbb{A}$ of dimension 2 over $\mathbb{K}$ (if $\mathbb{A}$ is quaternion), and a semi-linear involution θ of Δ (more exactly, θ is an involutive automorphism of Δ such that, whenever we have four collinear points $p_{1,2}$, p_3, p_4 in an arbitrary rank 2 residue of type A_2, viewed as a projective plane over $\mathbb{A}$, then the cross ratio $(p_1^\theta, p_2^\theta; p_3^\theta, p_4^\theta)$ is equal to $(p_1, p_2; p_3, p_4)^\sigma$, where σ is the Galois involution of the extension $\mathbb{A}/\mathbb{K}$), such that $\mathsf{F}_4(\mathbb{K},\mathbb{A})$ is the fix structure of θ in Δ. The type of Δ is called the *absolute type* of $\mathsf{F}_4(\mathbb{K},\mathbb{A})$, and the latter is referred to as the *relative building*. This construction is known as *Galois descent* and described by a generalisation of Witt index, nowadays called a *Tits index*, since it was introduced by Tits in [29]. Such an index consists of the type of the building Δ, furnished with some data among which most importantly the rank of the fix building. The other data are not important to us (and differ for classical and exceptional cases) and we refer to [29] for more details. However, for completeness and clarity, we will often provide this Tits index for some buildings we are dealing with (not only for the case of F_4 just described, but also for its residues), since it will help in understanding the arguments. A Tits index is usually also represented by a *Tits diagram*, which is the diagram of Δ furnished with some encircled nodes which geometrically represent the types of vertices fixed by θ. The number of circles is the rank of the relative building. Several nodes can be contained in the same circle when a flag is fixed, but not the vertices themselves of the flag. See also Appendix C of [32] for more explanation of this geometric interpretation of Galois descent.

For now we content ourselves with displaying the two Tits indices described in the previous paragraph for $\mathsf{F}_4(\mathbb{K}, \mathbb{A})$:

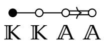
${}^2\mathsf{E}^2_{6,4}$		$\mathbb{A}$ quadratic field extension of $\mathbb{K}$
$\mathsf{E}^9_{7,4}$		$\mathbb{A}$ quaternion division algebra over $\mathbb{K}$

3 Embedding and generating ranks of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$

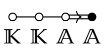
In this section we prove Theorem A.

The proof we present makes use of the so-called *extended equator geometries* and associated *tropics geometries*, which we introduce in Section 3.4 below. We first take a look at the generating and embedding ranks of some polar and dual polar spaces which will turn out to be isomorphic to these extended equator and tropics geometries.

3.1 Generation of some polar and dual polar spaces

Recall that we denote by $\mathsf{B}_{4,1}(\mathbb{K}, \mathbb{A})$  the (orthogonal) polar space of rank 4 with associated anisotropic form given by the norm form of the quadratic division algebra $\mathbb{A}$. The following theorem is a consequence of Corollary 8.7 of [30].

Proposition 3.1. *The embedding and generating rank of* $\mathsf{B}_{4,1}(\mathbb{K}, \mathbb{A})$ *is equal to* $8 + \dim_{\mathbb{K}} \mathbb{A}$.

We now turn to $\mathsf{B}_{4,4}(\mathbb{K}, \mathbb{A})$ . We begin with a generation result on dual polar spaces.

Lemma 3.2. *Let $\Delta = (X, \mathscr{L})$ be a dual polar space of rank $n \geq 3$ with the property that its quads are generated by two opposite lines. Then Δ is generated by 2^n points.*

Proof. Let Δ^* be the corresponding polar space of rank n. Pick a frame F in Δ^*, that is, a set of $2n$ points $\{p_{-n}, p_{-(n-1)}, \ldots, p_{-1}, p_1, p_2, \ldots, p_n\}$ such that each point p_i has exactly one opposite p_{-i} in F, $i \in \{-n, \ldots, -1, 1, \ldots, n\}$. Let S be the subspace of Δ generated by the set G of points corresponding to the maximal singular subspaces generated by n mutually collinear points of F. Note that $|G| = 2^n$.

We show by induction on $n - k \in \{1, \ldots, n\}$ that for each $k \in \{0, 1, \ldots, n-1\}$, each maximal singular subspace containing at least k collinear points of F belongs to S. If $n - k = 1$, then such maximal singular subspace belongs to a line of Δ having two points of G, and hence by definition of the subspace S, it also belongs to S. Now let $k < n - 1$. Without loss of generality, it suffices to show that all maximal singular subspaces containing $U := \langle p_1, p_2, \ldots, p_k \rangle$, belong to S. By induction, we know that all maximal singular subspaces containing $\langle U, p_{k+i} \rangle$ and $\langle U, p_{-k-i} \rangle$, for all $i \in \{1, 2, \ldots, n-k\}$, belong to S. Considering the residue of U, we may assume that $k = 0$ (and $n \geq 2$), and we have to show that, if all maximal singular subspaces containing some point of F belong to S, then all maximal singular subspaces do. We claim that, if each maximal singular subspace through one of two noncollinear points x and y belong to S, then so do all singular subspaces containing some point of $x^\perp \cap y^\perp$. Indeed, select $z \in x^\perp \cap y^\perp$

and let M be a maximal singular subspace through z. If $x^\perp \cap M = y^\perp \cap M$, then by the definition of subspace, we obtain $M \in S$. So we may assume that $W := M \cap x^\perp \cap y^\perp$ has codimension 2 in M. The residu of W is a quad in Δ with the property that all points on the lines corresponding to x and y belong to S. By assumption, also the point corresponding to M belongs to S and the claim follows. Now every point of Δ^* is collinear to two noncollinear points, say q_1 and q_{-1}, of $p_1^\perp \cap p_{-1}^\perp$. Applying the previous claim first with $(x, y) = (p_1, p_{-1})$ and then with $(x, y) = (q_1, q_{-1})$ shows the assertion. $\square$

Noting that, using Propositions 3.4.9, 3.4.11 and 3.4.13 of [32], the quads of $\mathsf{B}_{n,n}(\mathbb{K}, \mathbb{A})$ are precisely the quadrangles $\mathsf{C}_{2,1}(\mathbb{A}, \mathbb{K})$ which, by Proposition 5.9.6 of [32], are generated by two opposite lines if $\mathbb{A}$ is separable over $\mathbb{K}$, we immediately obtain:

Corollary 3.3. *The dual polar spaces* $\mathsf{B}_{n,n}(\mathbb{K}, \mathbb{A})$*, with* $\mathbb{A}$ *an associative separable quadratic division algebra over* $\mathbb{K}$*, have generating rank at most* 2^n*.*

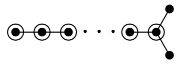
We will see in the next section that the generating rank is precisely 2^n by exhibiting a projective embedding in projective dimension $2^n - 1$.

3.2 Universal embeddings of some dual polar spaces

It is well known that the universal embedding rank of $\mathsf{B}_{n,n}(\mathbb{K}, \mathbb{A})$ is equal to 2^n for $\mathbb{A}$ commutative and separable over $\mathbb{K}$, see [36] and [9]. Nothing seems to be known for $\mathbb{A}$ quaternion. However, Corollary 3.3 implies that the embedding rank of $\mathsf{B}_{n,n}(\mathbb{K}, \mathbb{A})$, for $\mathbb{A}$ separable over $\mathbb{K}$, is at most 2^n. Exhibiting a projective embedding in $\mathbb{P}^{2^n-1}(\mathbb{A})$ would show at once that the embedding rank and generating rank of $\mathsf{B}_{n,n}(\mathbb{K}, \mathbb{A})$ is 2^n. That is exactly what we will do now.

Proposition 3.4. *For* $\mathbb{A}$ *separable over* $\mathbb{K}$*, the dual polar space* $\mathsf{B}_{n,n}(\mathbb{K}, \mathbb{A})$ *admits a full embedding in* $\mathbb{P}^{2^n-1}(\mathbb{A})$*.*

Proof. By the above references, we may assume that $\mathbb{A}$ is quaternion.

Following Tits [29], the absolute type of $\mathsf{B}_n(\mathbb{K}, \mathbb{A})$, $\mathbb{A}$ quaternion over $\mathbb{K}$, is D_{n+2} with corresponding Tits index ${}^1\mathsf{D}^{(1)}_{n+2,n}$  . Referring to Section 2.6, this means that $\mathsf{B}_n(\mathbb{K}, \mathbb{A})$ is obtained from $\mathsf{D}_{n+2}(\mathbb{L})$, with $\mathbb{L}$ a separable quadratic extension of $\mathbb{K}$ contained in $\mathbb{A}$ as a 2-dimensional algebra, by taking the fixed singular subspaces under a semi-linear involution θ which acts type-preservingly, fixes at least one singular subspace of any dimension d, $0 \le d \le n-1$, but does not fix any singular subspace of dimension $n+1$ (and hence none of dimension n either). To concretely obtain $\mathsf{B}_{n,n}(\mathbb{K}, \mathbb{A})$, one has to consider that involution θ in $\mathsf{D}_{n+2,n+2}(\mathbb{K})$, where it acts fixed point freely. The points of $\mathsf{B}_{n,n}(\mathbb{K}, \mathbb{A})$ are then the fixed lines; the lines of $\mathsf{B}_{n,n}(\mathbb{K}, \mathbb{A})$ are the sets of fixed lines of $\mathsf{D}_{n+2,n+2}(\mathbb{K})$ in a solid fixed under the action of θ. (This can be read off the Tits index.) Note that the companion field involution of θ is the restriction to $\mathbb{L}$ of the standard involution of $\mathbb{A}$ as quaternion algebra over $\mathbb{K}$.

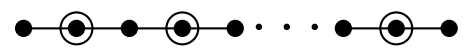
Now, according to Section 3 of [36], see also Proposition 5.3 in [34] for a completely geometric account which we will use below, the universal embedding of $\mathsf{D}_{n+2,n+2}(\mathbb{L})$ happens in $\mathbb{P}^{2^{n+1}-1}(\mathbb{L})$. By the universality of the embedding, the involution θ extends to $\mathbb{P}^{2^{n+1}-1}(\mathbb{L})$, and we denote the extension also by θ. Since θ^2 pointwise fixes $\mathsf{D}_{n+2,n+2}(\mathbb{L})$, it is the identity everywhere, so θ is an involution of $\mathbb{P}^{2^{n+1}-1}(\mathbb{L})$. We claim the following two things:

Claim 1: *The involution* θ *acts fixed point freely on* $\mathbb{P}^{2^{n+1}-1}(\mathbb{L})$*.*

Suppose for a contradiction that θ fixes a point x of $\mathbb{P}^{2^{n+1}-1}(\mathbb{L})$ and let L be a line of $\mathsf{D}_{n+2,n+2}(\mathbb{L})$ fixed by θ. Then the plane π generated by x and L is also fixed. Since θ is semi-linear, it induces a so-called Baer involution in it and hence its fixed point structure is a Baer subplane. But such a subplane has a point on each line, contradicting the fact that L does not contain fixed points.

Claim 2: *The lines of* $\mathsf{D}_{n+2,n+2}(\mathbb{L})$ *fixed by* θ *generate* $\mathbb{P}^{2^{n+1}-1}(\mathbb{L})$.

We use the inductive geometric construction of the universal embedding of $\mathsf{D}_{n+2,n+2}(\mathbb{L})$ as given in Proposition 5.3 in [34]. That construction implies that $\mathbb{P}^{2^{n+1}-1}(\mathbb{L})$ is generated by the two half spin subgeometries isomorphic to $\mathsf{D}_{n+1,n+1}(\mathbb{L})$ obtained from the residues of two noncollinear points in the corresponding polar space $\mathsf{D}_{n+2,1}(\mathbb{L})$. Taking for those points two points fixed by θ, we see that an inductive argument proves the claim if we check the smallest case $n = 1$. In that case, the set of fixed lines corresponds to a spread of $\mathbb{P}^{2^{n+1}-1}(\mathbb{L}) = \mathbb{P}^3(\mathbb{L})$ and so they generate the whole space trivially (as they even *fill* or *cover* the space). Claim 2 is proved.

Claim 1 now implies that θ induces the Tits index ${}^1\mathsf{A}^{(2)}_{2^{n+1}-1,2^n-1}$ 

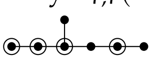
giving rise to the quaternion projective space $\mathbb{P}^{2^n-1}(\mathbb{A})$. Claim 2 implies that $\mathsf{B}_{n,n}(\mathbb{K},\mathbb{A})$ is fully embedded in and spans $\mathbb{P}^{2^n-1}(\mathbb{A})$. □

Remark 3.5. If $\mathbb{A}$ is separable over $\mathbb{K}$, but not quaternion, then the proof of Proposition 3.4 goes through, except that we do not consider a semi-linear involution, but a linear collineation if $\mathbb{A}$ is quadratic over $\mathbb{K}$ (not an involution if $\mathsf{char}\,\mathbb{K} = 2$), and a linear involution in $\mathsf{D}_{n+1}(\mathbb{K})$ if $\mathbb{K} = \mathbb{A}$. Noting that we only used the Tits indices as fix diagrams, we can use the appropriate fix diagrams to prove exactly the same claims and prove the proposition in these simpler cases. In fact, this amounts to the constructions given in [9] (for $\mathbb{A}$ quadratic over $\mathbb{K}$) and [36] (for $\mathbb{K} = \mathbb{A}$). Therefore we do not insist on it.

As noted above, this now implies:

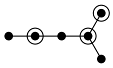
Corollary 3.6. *Both the embedding rank and generating rank of the dual polar spaces* $\mathsf{B}_{n,n}(\mathbb{K},\mathbb{A})$, *with* $\mathbb{A}$ *an associative separable quadratic division algebra over* $\mathbb{K}$, *are equal to* 2^n.

3.3 Some embeddings of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$

In the same way as for $\mathsf{B}_{n,n}(\mathbb{K},\mathbb{A})$ above, we will produce an embedding of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ in $\mathbb{P}^{27}(\mathbb{A})$, if $\mathbb{A}$ is quaternion over $\mathbb{K}$. The other cases for associative separable $\mathbb{A}$ over $\mathbb{K}$ can be done similarly, but are easier and have been considered elsewhere. Indeed, by Theorem 6.1 of [12], the embedding rank and generatng rank of $\mathsf{F}_{4,4}(\mathbb{K})$ is equal to 25. We will handle the case of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$, for $\mathbb{A}$ separable quadratic over $\mathbb{K}$, below after the quaternion case. Consider the Lie incidence geometry $\mathsf{E}_{7,7}(\mathbb{K})$ embedded in $\mathbb{P}^{55}(\mathbb{K})$ and let $\mathbb{A}$ be quaternion over $\mathbb{K}$. Using the Tits index $\mathsf{E}^9_{7,4}$ , we see that $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ arises from the fixed point structure of a semi-linear involution θ of $\mathsf{E}_{7,7}(\mathbb{L})$, where $\mathbb{L}$ is a quadratic extension of $\mathbb{K}$ stabilized under the standard involution of $\mathbb{A}$, by defining the points to be the fixed lines, and the lines of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ are identified with the fixed solids, in which the fixed lines form a spread.

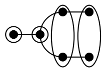
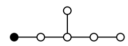
As in Claim 1 of the proof of Proposition 3.4, we deduce that θ extends to $\mathbb{P}^{55}(\mathbb{L})$ (since the embedding of $\mathsf{E}_{7,7}(\mathbb{L})$ in $\mathbb{P}^{55}(\mathbb{L})$ is universal [24, Theorem 4.1]) and θ does not have any fixed points in that projective space. Hence, again just like in the proof of Proposition 3.4, one obtains now an embedding of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ inside $\mathbb{P}^{27}(\mathbb{A})$. We claim that this embedding spans the full projective space.

Indeed, according to the Tits index (or the fix diagram), θ stabilizes two opposite symps, say ξ_1 and ξ_2. Let $i \in \{1,2\}$. The fixed point structure of θ in ξ_i conforms to the Tits index ${}^1\mathsf{D}^{(2)}_{6,3}$



Since every polar space of type $\mathsf{D}_{6,1}$ is embedded only in projective 11-space, this Tits index gives rise to an embedding of $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ in a subspace of $\mathbb{P}^5(\mathbb{A})$; but it has to generate it because $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ contains disjoint singular planes.

Now also the equator $E(\xi_1,\xi_2)$ (see Section 3.3 of [16]) is stabilized. The previous paragraph implies that ξ_i contains a *basis* lying on fixed lines, that is, a set of twelve points closed under the action of θ and such that each point is noncollinear to exactly one other point of the set. This in turn implies that the equator $E(\xi_1,\xi_2)$ contains a set of 16 fixed lines containing a basis, that is, a set of 32 points generating a 31-dimensional subspace of $\mathbb{P}^{55}(\mathbb{L})$. The claim now follows.

The case where $\mathbb{A}$ is a separable quadratic extension of $\mathbb{K}$ is handled with great similarity, now considering the Tits index ${}^2\mathsf{E}^2_{6,4}$  as follows. The Lie incidence geometry $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ arises from the fixed point structure of a semi-linear polarity ρ of $\mathsf{E}_{6,1}(\mathbb{A})$  by taking as points the absolute points of $\mathsf{E}_{6,1}(\mathbb{A})$ and as lines the absolute lines. The symps of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ are then obtained from the fixed 5-spaces of $\mathsf{E}_{6,1}(\mathbb{A})$, in which ρ logically induces a Hermitian polarity with absolute geometry a Hermitian polar space $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ (and those are indeed isomorphic to the symps of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$). In particular, we can consider two opposite fixed such 5-space W_1 and W_2. Now just like in the previous paragraphs, now using the equator $E(W_1,W_2)$ as defined in Section 3.2 of [16], we can show that the absolute points in $E(W_1,W_2)$ generate a 14-dimensional subspace, and so $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ generates $\mathbb{P}^{26}(\mathbb{L})$.

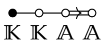
In conclusion we have shown:

Proposition 3.7. *There exists a full embedding of* $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ *in*

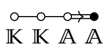
$$\begin{cases} \mathbb{P}^{25}(\mathbb{A}), & \text{if } \mathbb{A} = \mathbb{K} \text{ and } \mathsf{char}\,\mathbb{K} \neq 2, \\ \mathbb{P}^{26}(\mathbb{A}), & \text{if } \mathbb{A} \text{ is a separable quadratic extension of } \mathbb{K}, \\ \mathbb{P}^{27}(\mathbb{A}), & \text{if } \mathbb{A} \text{ is a quaternion algebra over } \mathbb{K}. \end{cases}$$

3.4 Generation of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$

Theorem A will be proved if we show that $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ is, as a geometry, generated by 26, 27 or 28 points, for $\mathbb{A}=\mathbb{K}$, $\mathbb{A}$ quadratic over $\mathbb{K}$, or $\mathbb{A}$ quaternion over $\mathbb{K}$, respectively. As already mentioned, the case $\mathbb{A}=\mathbb{K}$ with $\mathsf{char}\,\mathbb{K}\neq 2$, is handled by Theorem 6.1 of [11]. We now treat the two other cases. As already alluded to, this makes use of the so-called *extended equator geometry*, and the companion *tropics geometry*, which we introduce now. See [10], [20] and [23] for proofs.

Henceforth let $\mathbb{A}$ have dimension 2 or 4 over $\mathbb{K}$. Select two arbitrary opposite points p,q of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ and let $E(p,q)$ be the set, called *equator (with poles p,q)*, of points symplectic to both p and q. Let $\widehat{E}(p,q)$, called *extended equator*, be the union of all sets $E(x,y)$ with x and y opposite and contained in $E(p,q)$; it is independent of the choice of $p,q\in\widehat{E}(p,q)$. Let $\mathscr{L}(p,q)$ be the set of intersections of size at least 2 of $\widehat{E}(p,q)$ with symps of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$. Each such intersection is a hyperbolic line of the polar space $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$, hence obtained by the common perp of two opposite lines. The point-line geometry $\Delta(p,q) = (\widehat{E}(p,q), \mathscr{L}(p,q))$ is a polar space $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$ 

.

Let $\widehat{T}(p,q)$ be the set of points of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ collinear to at least two points of $\widehat{E}$ (each such point x is collinear to exactly the points of a maximal singular subspace $\beta(x)$ of $\Delta(p,q)$). Let $\mathscr{M}(p,q)$ be the set of lines of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ entirely contained in $\widehat{T}(p,q)$. Then the point-line geometry $\Omega = (\widehat{T}(p,q), \mathscr{M}(p,q))$, called *tropics geometry*, is a dual polar space $\mathsf{B}_{4,4}(\mathbb{K},\mathbb{A})$ 

. The correspondence is quite neat: The mapping β is bijective onto the set of maximal subspaces; two points x,y of Ω are collinear if and only if $\beta(x)$ and $\beta(y)$ intersect in a plane of Δ, they are symplectic if and only if $\beta(x)$ and $\beta(y)$ intersect in a line of Δ, they are special if and only if $\beta(x)$ and $\beta(y)$ intersect in a point (which equals $x \bowtie y$) of Δ and they are opposite if and only if $\beta(x)$ and $\beta(y)$ are disjoint. Moreover, if $x,y \in \widehat{T}(p,q)$ are collinear, then β is a bijection from the set of points of the line xy to the set of planes of Δ through the line $\beta(x) \cap \beta(y)$. Also, a point of $\widehat{T}(p,q)$ and a point of $\widehat{E}(p,q)$ are either collinear or special, but never symplectic or opposite.

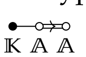
We fix an extended equator geometry $\widehat{E}$ and its companion tropics geometry $\widehat{T}$ for the rest of this section (and we forget p and q). We denote by Δ and Ω the corresponding point-line geometries as introduced above. Let $\Xi(\widehat{E})$ be the set of all symps containing some point, and hence some line, of Δ, and let $\widehat{E}_{\perp\!\!\perp}$ be the union of all those, viewed as sets of points. Likewise, let $\mathscr{L}(\widehat{T})$ be the set of all lines containing some point of $\widehat{T}$, and let $\widehat{T}_{\perp}$ be the union of all those lines, viewed as sets of points. We have the following observation.

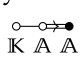
Lemma 3.8. *The inclusion $\widehat{E}_{\perp\!\!\perp} \subseteq \widehat{T}_{\perp}$ is always valid.*

Proof. Let ξ be a symp containing some point x of $\widehat{E}$. Let y be a point of $\widehat{E}$ opposite x. There is a symp containing y which intersects ξ nontrivially. Hence ξ contains a line h of Δ. Then $h^{\perp} \subseteq \xi$ contains at least one line (it has the structure of $\mathsf{C}_{2,1}(\mathbb{A},\mathbb{K})$), and so every point of ξ is collinear to (lots of) points of $h^{\perp} \subseteq \widehat{T}$. □

The converse of Lemma 3.8 is not true, but nevertheless one can show the following:

Lemma 3.9. *The set $\widehat{T}_{\perp}$ is contained in the subspace generated by $\widehat{E}_{\perp\!\!\perp}$.*

Proof. Let p be an arbitrary point of $\widehat{T}$. Then $\beta(p) \subseteq \widehat{E}$ is a solid of Δ. Each line h of $\beta(p)$ defines a symp $\xi(h)$ containing p. We claim that the planar line pencils of $\beta(p)$ correspond to the set of symps containing a fixed plane through p. Indeed, let c be a point of $\beta(p)$ and π a plane of Δ in $\beta(p)$ containing c and let $\Pi(c,\pi)$ be the corresponding line pencil. Let L be the line of $\widehat{T}$ with the property that for each point $x \in L$ the solid $\beta(x)$ contains π. Then each symp defined by a member of $\Pi(c,\pi)$ contains c and L, and hence contains the plane α generated by c and L. Conversely, a symp containing α contains c and hence contains a line h of $\widehat{E}$. Since the points of L are in a symplecton with h, they are collinear to h an so h belongs to $\Pi(c\pi)$. The claim is proved. Consequently the set $\Xi(p)$ of symps $\xi(h)$, with h running through the set of lines of $\beta(p)$, corresponds to a hyperbolic quadric $\mathscr{H}$ of Witt index 3 fully embedded in the polar space $\Pi \cong \mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$ 

corresponding to $\mathsf{Res}(p) \cong \mathsf{B}_{3,3}(\mathbb{K},\mathbb{A})$ 

.

We claim that every plane α through p contains at least two distinct lines of members of $\Xi(p)$. Indeed, α corresponds to a line L of Π, whereas $\mathscr{H}$ can be considered a full subquadric of Π.

We have to show that L is contained in at least two planes intersecting $\mathscr{H}$ nontrivially, and we may clearly assume that L is disjoint from $\mathscr{H}$. Now $\mathscr{H}$ contains at least two opposite planes, and these contain distinct points collinear to L. The claim follows.

Our claim implies that every plane through p is fully contained in the subspace generated by the symps $\xi(h)$, with h as above. Since those are contained in $\widehat{E}_{\perp\!\!\perp}$, the assertion follows. $\square$

Let T be a set of 16 points generating $\widehat{T}$ as a dual polar space isomorphic to $\mathsf{B}_{4,4}(\mathbb{K}, \mathbb{A})$, cf. Corollary 3.3. Let E be a set of $8 + \dim_{\mathbb{K}} \mathbb{A}$ points generating $\widehat{E}$ as a polar space, cf. Proposition 3.1. As explained in Section 6 of [12] $T \cup E$ generates the subspace generated by $\widehat{E}_{\perp\!\!\perp}$, and hence, by Lemma 3.9, it also generates the subspace generated by $\widehat{T}_{\perp}$.

The following proposition will imply Theorem A for $\mathbb{A}$ not quaternion.

Proposition 3.10. *The set $\widehat{T}_{\perp}$, and hence $T \cup E$, generates a subspace $\widehat{H}$ which is either a geometric hyperplane, or coincides with the whole of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$.*

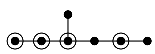
Proof. Let ξ be any symp of $\mathsf{F}_{4,4}(\mathbb{K}, A)$. We claim that $H \cap \xi$ contains a geometric hyperplane of ξ. If ξ contains some point of $\widehat{E}$, this is trivial. Suppose ξ is close to some point x of $\widehat{E}$. Then there exists a symp ζ containing x and intersecting ξ in a plane α. Now, $\widehat{T} \cap \zeta$ is the common perp of the points of $\widehat{E} \cap \zeta$. It follows that $\widehat{T} \cap \zeta$ is a subhyperplane of ζ, implying that α contains a point $z \in \widehat{T}$. So ξ contains all points of ξ collinear to $z \in \xi$, which is a hyperplane of ξ.

Finally suppose that all points of $\widehat{E}$ are far from ξ. Select $x \in \widehat{E}$; let y be the unique point of ξ symplectic to x and let ζ be the symp defined by x and y. Then $\zeta \in \Xi(\widehat{E})$, and hence $H_y := y^{\perp} \cap \widehat{T} \cap \zeta$ is a hyperplane of $\widehat{T} \cap \zeta$. This implies that y and H_y generate a subhyperplane of ζ. Using Fact 2.5, the set of points of ξ collinear to some point of H_y constitutes a subhyperplane of ξ all points of which are collinear to y, and which is contained in $\widehat{H}$ (as each point is collinear to some point of $\widehat{T}$). Repeating this argument with $x' \notin \zeta$, $x' \perp\!\!\perp x$, such that $\xi(x, x')$ is opposite ξ, we obtain a second subhyperplane of ξ all points of which are this time collinear to y', which is not collinear to y (as x and x' are not collinear), and which is contained in $\widehat{H}$. Hence the two subhyperplanes do not coincide and therefore generate a geometric hyperplane of ξ, contained in $\widehat{H}$. $\square$

Proposition 3.11. *If $\mathbb{A} = \mathbb{K}$ and $\mathsf{char}\,\mathbb{K} \neq 2$, then the embedding and generating ranks of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ are both equal to 26; if $\mathbb{A}$ is a separable quadratic extension of $\mathbb{K}$, then the embedding and generating ranks of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ are both equal to 27.*

Proof. Proposition 3.10 implies, by Theorem 2.2 in [18], that the generating rank is at most 26 and 27, respectively; Proposition 3.7 then implies the assertions. $\square$

In the quaternionic case, the previous results only imply that the generating and embeddings ranks belong to $\{28, 29\}$, since we have an embedding in projective 27-space, and we know that the geometry is generated by at most 29 points. Our next objective is to show that $\widehat{H}$, as defined in Proposition 3.10, coincides with $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$, if $\mathbb{A} = \mathbb{H}$ is a quaternion division algebra over $\mathbb{K}$.

To that end, we use the construction of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{H})$ via Galois descent. Let $\mathbb{L}$ be a separable quadratic extension of $\mathbb{K}$ contained in $\mathbb{H}$ such that the standard involution σ of $\mathbb{H}$ acts as the Galois involution on $\mathbb{L}/\mathbb{K}$. Then in view of the Tits index $\mathsf{E}^{9}_{7,4}$ , there exists a σ-semilinear involution θ on $\mathsf{E}_{7,7}(\mathbb{L})$ such that its fixed lines can be identified with the points of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{H})$, the fixed 3-spaces with the lines of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{H})$, the fixed maximal 5-spaces with the planes of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{H})$, and the fixed symps with the symps of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{H})$. First we interpret extended equators of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{H})$ in $\mathsf{E}_{7,7}(\mathbb{K})$.

Lemma 3.12. *An extended equator geometry of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{H})$ corresponds to the set of lines intersecting two opposite symps ξ and ξ^θ and inducing in each a laxly embedded subquadric isomorphic to $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{H})$. This induced set in ξ generates ξ as a subspace of itself.*

Proof. Let p, q be opposite points in $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{H})$ and let L_p, L_q be the corresponding lines in $\mathsf{E}_{7,7}(\mathbb{L})$. Since θ is a Galois automorphism of algebraic groups, the lines L_p and L_q are opposite. Let $x \in L_p$ be a point, and let $y \in L_q$ be the unique point at distance 2 from x. Denote by ξ the symp containing x and y. Let r be a point of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{H})$ symplectic to both p, q, and let L_r be the corresponding line in $\mathsf{E}_{7,7}(\mathbb{L})$. We claim that ξ contains a unique point of L_r. Indeed, by the definition of r there exist symps ζ and ν containing L_p and L_q, respectively, and intersecting in L_r. The point x is collinear to a unique point $z \in L_r$, and z is collinear to a unique point of L_q. Since y is the unique point on L_q not opposite x, we necessarily have $y \perp z$, and so $y \in \xi$. The claim is proved. Likewise, L_r contains $y^\theta \in \xi^\theta$.

Interpreting the previous claim, we have shown that $\widehat{E}$ laxly embeds in ξ. This proves the first assertion. Define the following involution $\theta_\xi : \xi \to \xi : u \mapsto u^{\theta_\xi}$, where u^{θ_ξ} is the unique point of ξ collinear to u^θ. Since L_p and L_q are opposite, the symps ξ and ξ^θ are opposite and so θ_ξ is well defined. Moreover, the fixed point set of θ_ξ corresponds precisely to $\widehat{E}$. Clearly, θ_ξ is a semi-linear involution, and hence the fixed point set is a fully embedded quadric $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{H})$ in a subspace over $\mathbb{K}$. Since the only full embeddings of $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{H})$ occur in dimension 11 (by [30, Theorem 8.6]), the second assertion follows. □

From now one we fix a pair of opposite symps ξ, ξ^θ such that the fixed lines intersecting both correspond to the points of an extended equator $\widehat{E}$. We denote the subspace generated by $\widehat{E}_{\perp\!\!\!\perp}$ by $\widehat{H}$. Each point $x \in \widehat{E}$ corresponds to a fixed line L_x intersecting both ξ and ξ^θ in points which we denote by x_ξ and x^θ_ξ, respectively.

We say that a line is *far* from a symp if every point has a unique collinear point in the symp. Note that the latter point cannot be the same for all points of L as otherwise a point of the symp at distance 2 from that point is at distance 3 from all points of L, contradicting Corollary 2.7.

Lemma 3.13. *A line L is far from a symp ζ if and only if at least two distinct points of L are collinear to distinct unique points of the symp.*

Proof. The "only if" part being obvious, we suppose for a contradiction that some point x of L is close to ζ, say collinear with the 5-space U, and that $x_1, x_2 \in L$ are such that $x_i \perp x_i' \in \zeta$, $i = 1, 2$, with $x_1 \neq x_2$ and $x_1' \neq x_2'$ unique. If $x_1 \in U$, then $L \perp x_1' \neq x_2'$, a contradiction to the uniqueness of x_2'. Hence $x_1' \notin U$ and so there is a symp through x_1' and x. That symp also contains L and a 4-subspace of U. Within that symp, the point x_1 is now collinear to at least a 3-space of U, the final contradiction. □

Lemma 3.14. *If a line L is far from a symp ζ, then it is contained in a unique symp ζ' intersecting ζ in a line. Moreover, every symp locally opposite ζ' at L is globally opposite ξ.*

Proof. Let x, y be two points on L, and let x', y' be the corresponding points in ζ (so $x \perp x'$ and $y \perp y'$). By the note above, we may assume $x' \neq y'$. We show that $x' \perp y'$. Indeed, if not, then y' is opposite x by Fact 2.6, contradicting $x \perp y \perp y'$. Hence x' and y' span a line L'. Now the symp ζ' through x and y' contains L and L'. If $\zeta \cap \zeta'$ were a 5-space, then x would be collinear to a 5-space of ζ, a contradiction. The last assertion follows from a translation to $\mathsf{E}_{7,1}(\mathbb{K})$, where symps become points and lines become symps. The assertion is then equivalent to the well known fact in long root geometries that, if two symps intersect precisely in one point x, and the symps are locally opposite at x, then every point of one symp not collinear to x, is opposite each point of the other symp not collinear to x. $\square$

Lemma 3.15. *Let ζ be a fixed symp disjoint from $\xi \cup \xi^\theta$. Suppose ζ is not opposite ξ. Then each line L of ζ disjoint from the unique 5-space $U \subseteq \zeta$ that is contained in a symp together with some 5-space of ξ, is far from ξ.*

Proof. Each point of L is collinear to exactly one point of ξ, by Fact 2.8(v). $\square$

Lemma 3.16. *A symp of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{H})$ corresponding to a fixed symp ζ of $\mathsf{E}_{7,7}(\mathbb{L})$ opposite ξ entirely belongs to $\widehat{H}$.*

Proof. Let x be a point of $\widehat{E}$. There is a unique line through x_ξ intersecting ζ in a unique point y. Hence x_ξ^θ is collinear to a unique point y^θ of ζ. Lemma 3.13 asserts that L_x is far from ζ and hence, by Lemma 3.14, it is contained in a(n automatically fixed) symp intersecting ζ in a (necessarily fixed) line M_x, which contains both y and y^θ. Now M_x corresponds to a point of $\widehat{H}$ since it is contained in a symp through some point of $\widehat{E}$.

Also, the set of points x_ξ, as x runs through $\widehat{E}$, generates ξ (as a subspace); hence, by Fact 2.8(v), the points y generated ζ, and so the points of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{H})$ corresponding to the lines M_x generate the symp corresponding to ζ. $\square$

Lemma 3.17. *The subspace $\widehat{H}$ coincides with $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{H})$.*

Proof. We first claim that, if a point x of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{H})$ corresponds to a line L of $\mathsf{E}_{7,7}(\mathbb{L})$ far from ξ, then it belongs to $\widehat{H}$. Indeed, local opposition at L in $\mathsf{E}_{7,7}(\mathbb{L})$ corresponds to local opposition at x in $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{H})$. This implies that Lemma 3.14 yields a symp of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{H})$ through x corresponding to a symp of $\mathsf{E}_{7,7}(\mathbb{L})$ opposite ξ, and hence completely contained in $\widehat{H}$. The claim is proved.

Now suppose the line L is not far from ξ. We can include x in a symp disjoint from $\widehat{E}$; hence L is contained in a symp ζ of $\mathsf{E}_{7,7}(\mathbb{L})$ disjoint from ξ. If ζ is opposite ξ, then we are done by the first claim. Suppose ζ is special to ξ. Then Lemma 3.15 yields a (fixed) 5-space U of ζ with the property that every fixed line outside U is far from ξ. The point of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{H})$ corresponding to such lines are contained in $\widehat{H}$, by the above claim. These points clearly generate the symp of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{H})$ corresponding to ζ (as they are the points not contained in the plane corresponding to U). Hence x belongs to $\widehat{H}$ and the lemma follows. $\square$

This now implies that $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{H})$ is generated by 28 points, and so Proposition 3.7 implies:

Proposition 3.18. *The embedding and generating ranks of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{H})$ are both equal to* 28.

This completes the proof of Theorem A.

Remark 3.19. It is routine to check that Proposition 3.10 is also valid in the octonionic case ($\mathbb{A} = \mathbb{O}$ nonassociative). Moreover, by [22], $\mathsf{C}_{3,1}(\mathbb{O}, \mathbb{K})$ is generated by the common perp of two opposite points, together with one other well-chosen point. Hence, if we have two symplectic points in E, then we may substitute these by one point in the symp generated by the two symplectic points. Since we can clearly choose E in such a way that we have 16 pairs of symplectic points, we can replace this by a set E' of 8 points in such a way that $T \cup E'$ generates $\widehat{T}_\perp$. Using Theorem 2.2 of [18], we then need at most one more point to generate the whole of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{O})$. Hence $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{O})$ is generated by at most 25 points, which is slightly surprising. However, if it were exactly 25, then it would remarkably but nicely complete our series of $26, 27, 28$ points for $\mathbb{A}$ dimension $1, 2, 4$ over $\mathbb{K}$, respectively, at the "wrong" side!

Remark 3.20. The inseparable case, that is, the case of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$, where $\mathbb{A}$ is an inseparable field extension of $\mathbb{K}$ in characteristic 2, is a true exception to our theorem. For instance, if $\mathbb{K} = \mathbb{A}^2$, then $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A}) \cong \mathsf{F}_{4,1}(\mathbb{K}, \mathbb{K})$ and so admits an embedding in $\mathsf{PG}(51, \mathbb{K})$, namely, its Weyl embedding, see [3]. The universal embedding always exists, however, but there is no explicit (geometric) description available.

4 Abstract dual polar varieties

In this section we introduce abstract dual polar varieties and we will prove Theorem C.

4.1 Axioms

The usual axioms for an abstract Veronesean variety, as considered for the first time in full generality in the split case in [27], require a (spanning) point set X in some projective space, and a number of subspaces, called host spaces, intersecting X in a quadric. In our case, the quadrics are replaced by Hermitian varieties and/or symplectic polar spaces. In the latter case, the intersection of the ambient projective subspace with X does not determine the polar space, since there are lines of the subspace which are not lines of the polar space. Hence in the set-up of the present paper, we have to furnish every host space with a line set of a polar space. This also allows for host spaces to admit several polar spaces. In order not to overload the notation, we will denote the polar space in question by a greek letter, usually ξ, and think of it as a pair $(\mathbb{P}(\xi), \mathscr{L}(\xi))$, where $\mathbb{P}(\xi)$ is the ambient subspace (the host space), and $\mathscr{L}$ is the set of lines of ξ. In the Hermitian case, $\mathscr{L}$ is completely determined by $\mathbb{P}(\xi) \cap X$. We call points of ξ the points of $\mathbb{P}(\xi) \cap X$.

Denote with $\mathbb{A}$ an associative division algebra (a skew field). Suppose $N \in \mathbb{N} \cup \{\infty\}$, and denote with $\mathbb{P}^N(\mathbb{A})$ an N-dimensional projective space over $\mathbb{A}$. Let X be a spanning point set of $\mathbb{P}^N(\mathbb{A})$, and let Ξ be a nonempty collection of embedded generalized quadrangles viewed (as explained above) as pairs $\xi = (\mathbb{P}(\xi), \mathscr{L}(\xi))$ with $\mathbb{P}(\xi)$ a 3-dimensional subspace of $\mathbb{P}^N(\mathbb{A})$ and $\mathscr{L}(\xi)$ the line set of a thick generalized quadrangle with point set $\mathbb{P}(\xi) \cap X$ which is fully embedded in $\mathbb{P}(\xi)$. For each point x of $\xi \in \Xi$, denote with $T_x(\xi)$ the tangent space at x to ξ (this is generated by the members of $\mathscr{L}(\xi)$ containing x), and let T_x be the subspace of $\mathbb{P}^N(\mathbb{A})$ generated by all the subspaces $T_x(\xi)$ for ξ running through all members of Ξ containing x. We assume *connectivity*, that is, the graph on the points of X, adjacent when contained in a common member of Ξ, is connected.

We assume that the following axioms hold.

(DP1) For any two points $x, y \in X$, either there exists an element $\xi \in \Xi$ with $x, y \in \mathbb{P}(\xi)$, or $T_x \cap T_y = \emptyset$. The latter occurs at least once.
(DP2) For $\xi_1, \xi_2 \in \Xi$ with $\xi_1 \neq \xi_2$, we have $\mathbb{P}(\xi_1) \cap \mathbb{P}(\xi_2) \subseteq X$.
(DP3) For each $x \in X$, the subspace T_x is at most 3-dimensional.

We call (X, Ξ) an *abstract dual polar variety*.

We denote by $\mathscr{L}$ the union of all $\mathscr{L}(\xi)$ for ξ running through Ξ. By our connectivity assumption, the geometry $(X, \mathscr{L})$ itself is connected. If two points $x, y \in X$ are contained in a common member of $\mathscr{L}$, then we write $x \perp y$ and call x and y *collinear* (if not collinear, they are *noncollinear*). In accordance with the first paragraph, we often identify the member $\xi \in \Xi$ with its point or line set, using expressions like "a point x of ξ" when we mean $x \in \mathbb{P}(\xi) \cap X$, or "a line L of ξ" when we mean $L \in \mathscr{L}(\xi)$, and we denote $x \in \xi$ or $L \subseteq \xi$, respectively.

4.2 Convexity

Our first goal is to show a convexity property, that is, if for $x, y \in X$, we have $T_x \cap T_y \neq \emptyset$, then the member $\xi(x, y)$ of Ξ containing x and y is unique as soon as x and y do not belong to the same member of $\mathscr{L}$. Note that this is immediate from (DP2) if the generalized quadrangle ξ is not symplectic. We proceed with a series of lemmas.

Lemma 4.1. *If a point $x \in X$ is contained in the intersection $\mathbb{P}(\xi_1) \cap \mathbb{P}(\xi_2)$, for $\xi_1, \xi_2 \in \Xi$, $\xi_1 \neq \xi_2$, then there is a line $L \in \mathscr{L}$ through x which is contained in $\mathbb{P}(\xi_1) \cap \mathbb{P}(\xi_2)$, and L belongs to both $\mathscr{L}(\xi_1)$ and $\mathscr{L}(\xi_2)$.*

Proof. Note that $x \in X$. The planes $T_x(\xi_1)$ and $T_x(\xi_2)$ are contained in the 3-space T_x and hence have a line in common. This line is contained in $\mathbb{P}(\xi_1) \cap \mathbb{P}(\xi_2)$, and hence in X. Since it is contained in $T_x(\xi_1)$, it actually belongs to $\mathscr{L}(\xi_1)$. Similarly for ξ_2. □

Lemma 4.2. *The diameter of $(X, \mathscr{L})$ is equal to* 3.

Proof. Suppose for a contradiction that x and y are two points of X at distance 4 from each other and let $x \perp u \perp v \perp w \perp y$, $u, v, w \in X$. Since $u \in T_x \cap T_v$, there exists $\xi_x \in \Xi$ containing x and v; likewise there exists $\xi_y \in \Xi$ containing v and y. By Lemma 4.1, ξ_x and ξ_y share a line $L \in \mathscr{L}$. Then there are points $x' \perp x$ and $y' \perp y$ on L in ξ_x and ξ_y, respectively, so that $x \perp x' \perp y' \perp y$, contradicting $d(x, y) = 4$. □

Lemma 4.3. *Let $\xi_1, \xi_2 \in \Xi$. If $\xi_1 \cap \xi_2$ is a plane π, then there is some $x \in \pi$ such that $\pi = T_x(\xi_1) = T_x(\xi_2)$.*

Proof. Let y be an arbitrary point in $\xi_1 \cap \xi_2$. By Lemma 4.1 $T_y(\xi_1) \cap T_y(\xi_2)$ contains a line L_y which is a line in both ξ_1 and ξ_2. Let z be a point in $\xi_1 \cap \xi_2 \setminus L_y$. We similarly obtain a line L_z. The unique intersection point $x = L_y \cap L_z$ is contained in two lines in both ξ_1 and ξ_2, hence $\pi = T_x(\xi_1) = T_x(\xi_2)$. □

Lemma 4.4. *Let ξ_1 and ξ_2 be arbitrary distinct members of Ξ. If $\mathbb{P}(\xi_1) = \mathbb{P}(\xi_2)$, then there is a unique line L of $\mathbb{P}(\xi_1)$, which automatically belongs to $\mathscr{L}(\xi_1) \cap \mathscr{L}(\xi_2)$, such that:*

(i) For each $x \in L$, the subspace $T_x \cap \mathbb{P}(\xi_1)$ is a plane, which coincides with $T_x(\xi_1)$ and $T_x(\xi_2)$.
(ii) For each $x \in \mathbb{P}(\xi_1) \setminus L$, the set T_x coincides with $\mathbb{P}(\xi_1)$.

Also, for $\xi' \in \Xi$ with $\mathbb{P}(\xi') \cap \mathbb{P}(\xi_1) \neq \emptyset$, one has $\mathbb{P}(\xi') \cap \mathbb{P}(\xi_1) = L$.

Proof. Note that this only occurs when $\mathbb{P}(\xi_1) = X(\xi_1)$. Moreover, if for all $x \in \mathbb{P}(\xi_1)$, one would have that $T_x \subseteq \mathbb{P}(\xi_1)$, then every ξ' intersecting ξ_1 would have the same point set $\mathbb{P}(\xi_1)$, connectedness then implies that $X = \mathbb{P}(\xi_1)$, a contradiction to Lemma 4.2.

We can hence find some point $x \in \mathbb{P}(\xi_1)$ with $T_x \neq T_x(\xi_1) = T_x \cap \mathbb{P}(\xi_1)$. Note that $T_x(\xi_2)$ is contained in $T_x \cap \mathbb{P}(\xi_1)$, implying that $T_x(\xi_1) = T_x(\xi_2)$. Also, T_x is 3-dimensional, and so there exists some $\xi \in \Xi$ such that $x \in \mathbb{P}(\xi) \neq \mathbb{P}(\xi_1)$. By Lemma 4.1, there is a line L contained in $T_x(\xi) \cap \mathbb{P}(\xi_1)$. If there is some point y of L for which $T_y \subseteq \mathbb{P}(\xi_1)$, then clearly $\mathbb{P}(\xi) \cap \mathbb{P}(\xi_1) = T_y(\xi)$. Now Lemma 4.3 implies $T_y(\xi) = T_y(\xi_1)$. For each point $z \in T_y(\xi) \setminus L$, we have $T_z(\xi)$ is not contained in $\mathbb{P}(\xi_1)$; consequently $T_z(\xi_1) = T_z(\xi_2)$ (as otherwise T_z would be at least 4-dimensional). However, it now follows directly from the Remark after Application 1 of Section 3 of [17] that $\xi_1 = \xi_2$, a contradiction. Hence for each $y \in L$ the tangent space T_y is not contained in $\mathbb{P}(\xi_1)$; so $T_y(\xi_1) = T_y(\xi_2)$. This shows (i).

Now the same remark in [17] shows (ii) and the uniqueness of L. The last assertion follows from the uniqueness of L. □

Lemma 4.5. *If $\xi_1, \xi_2 \in \Xi$, and $\xi_1 \neq \xi_2$, then $\mathbb{P}(\xi_1) \neq \mathbb{P}(\xi_2)$.*

Proof. Suppose for a contradiction that $\mathbb{P}(\xi_1) = \mathbb{P}(\xi_2)$. Let L be the line obtained in Lemma 4.4, and let $x \in L$. Since X contains two points with disjoint tangent spaces (by (DP1)), there exist points of X outside $\mathbb{P}(\xi_1)$. By connectivity of $(X, \mathscr{L})$, there exists $\xi' \in \Xi$ intersecting $\mathbb{P}(\xi_1)$ nontrivially, and then the last assertion of Lemma 4.4 yields $\mathbb{P}(\xi') \cap \mathbb{P}(\xi_1) = L$. Let $x' \in T_x(\xi') \setminus L$ (with $x' \in X$) and $x_1 \in T_x(\xi) \setminus L$. Then $x \in T'_x \cap T_{x_1}$, so there exists some $\xi'' \in \Xi$ with $x_1, x' \in \mathbb{P}(\xi'')$, but $x_1 \notin L$, a contradiction to the last assertion of Lemma 4.4. □

Remark 4.6. From now on we can identify ξ with its 3-dimensional subspace $\mathbb{P}(\xi)$ in $\mathbb{P}^N(\mathbb{A})$, so we drop the notation $\mathbb{P}(\xi)$.

Corollary 4.7. *Take $\xi_1, \xi_2 \in \Xi$. If a line L is contained in $\xi_1 \cap \xi_2$, then it belongs to $\mathscr{L}(\xi_1)$ if, and only if, it belongs to $\mathscr{L}(\xi_2)$.*

Proof. Immediate from Lemmas 4.1, 4.3 and 4.5. □

Definition 4.8. The distance $\delta(x, y)$, $x, y \in X$, is the distance in the collinearity graph of $(X, \mathscr{L})$, that is, the graph with vertices the members of X, adjacent when collinear.

Lemma 4.9. *Let x and y be two points of X with $\delta(x, y) > 2$, then $T_x \cap T_y = \emptyset$.*

Proof. This follows from (DP1) since symps have diameter 2. □

Lemma 4.10. *Let x and y be two points of X with $\delta(x, y) = 3$, and let z be collinear to x and at distance 2 of y. For any $\xi \in \Xi$ containing z and y, the point z is the unique point in ξ collinear to x. Moreover, every point of ξ that is noncollinear to z is at distance 3 from x.*

Proof. Suppose for a contradiction that x is collinear to some point z' of ξ, different from z. Then T_x intersects ξ in at least a line, and hence T_y in a point, a contradiction to Lemma 4.9.

Next, let y' be a point of ξ that is noncollinear to z. It is clear that $\delta(x, y') \leq \delta(x, z) + \delta(z, y') = 3$. Suppose for a contradiction that $\delta(x, y') = 2$. Then there exists some $\xi' \in \Xi$ that contains both x and y'. By Lemma 4.1, the intersection $\xi \cap \xi'$ is a line of X. The point x is collinear to a point of this line, which, by the previous argument, equals z. But then z is collinear to y', a contradiction. □

Lemma 4.11. *Let x and y be two points of X with $\delta(x, y) = 3$, and let $x \perp z_1 \perp z_2 \perp y$ be a path in X such that there is some $\xi \in \Xi$ which contains z_1, z_2 and y. If $\xi' \in \Xi$ contains x and z_2, then it also contains z_1.*

Proof. Let $\xi' \in \Xi$ contain x and z_2. By Lemma 4.1, there is a line L of X through z_2 that is contained in $\xi \cap \xi'$. Both x and L are contained in the generalized quadrangle ξ', so x is collinear to some point of L. It follows from Lemma 4.10 that this point is z_1, implying that z_1 is contained in ξ'. □

Lemma 4.12. *Let y be a point of X. The points at distance at most 2 from y form a subspace of X.*

Proof. Let L be a line of X containing a point z_1 at distance 2 from y, and suppose that L contains a point x at distance 3 from y. Since $T_{z_1} \cap T_y \neq \emptyset$, there is some member ξ_1 of Ξ that contains y and z_1. Let z_2 be point of X in ξ_1 that is collinear to both z_1 and y. As before, one can take an element $\xi \in \Xi$ that contains x and z_2. By Lemma 4.11, we find $z_1 \in \xi$, and hence also $L \subset \xi$. The assertion now follows from Lemma 4.10. □

Lemma 4.13. *Let ξ be an arbitrary member of Ξ. Suppose that (A) is some property of ordered pairs of non-collinear points such that, whenever (A) holds for the ordered pair (p,q) of noncollinear points of ξ, it also holds for all ordered pairs (p,q') and (p',q) of noncollinear points with $p \perp p'$ and $q \perp q'$. If (A) holds for some ordered pair of two noncollinear points of ξ, it holds for all ordered pairs of noncollinear points of ξ.*

Proof. This follows from the connectivity of the geometry far from a point in any generalized quadrangle, see [5] or [28, Lemma 7.5.2]. □

Lemma 4.14. *Let $\xi \in \Xi$ be arbitrary and suppose that there is an ordered pair of noncollinear points (y,z) in ξ for which*

(A) *there exists a point $x \perp z$ with $\delta(x,y) = 3$.*

Then Property (A) *holds every ordered pair of noncollinear points of ξ.*

Proof. By Lemma 4.13, it suffices to prove this for ordered pairs of noncollinear points (y',z) and (y,z') with y' collinear to y and z' collinear to z. For such point y', this immediately follows from Lemma 4.12. We prove it for z'. To that end, let z_2 be the unique point on zz' collinear to y, and take an element $\xi_2 \in \Xi$ that contains x and z_2. By Lemma 4.11, we have that $z \in \xi_2$, implying that $z' \in \xi_2$. By Lemma 4.10, and any point on ξ_2 noncollinear to z_2 is at distance 3 from y. So by taking x' to be a point of ξ_2 collinear with z' not on zz', we find $\delta(x',z') = 1$ and $\delta(z',y) = 3$. The lemma is proved. □

Lemma 4.15. *For any two points (y,z) of X at distance two, there exists a point x with $\delta(x,z) = 1$ and $\delta(x,y) = 3$.*

Proof. By Lemma 4.2, there exist such points y and z. Let ξ be an element of Ξ containing y and z. By Lemma 4.14, the claim holds for any two noncollinear points in ξ. Let ξ' be an element of Ξ that intersects ξ in at least a line L, we prove that the claim also holds for all pairs in ξ'. To that end, take some point y_1 on L and some z_1 on $\xi \setminus \xi \cap \xi'$ noncollinear to y_1. Let w be the point on L collinear to z_1. There is some point x_1 collinear to z_1 and at distance 3 from y_1. Let ξ'' be an element that of Ξ that contains x_1 and w, then ξ'' intersects ξ' in a point, and, by Lemma 4.1, in at least a line. The point x_1 is hence collinear to some point of ξ'. We find that the claim also holds for noncollinear pointpairs of ξ'. We can conclude the proof using connectedness. □

Lemma 4.16. *For any two points y and z at distance two, there is a unique element of Ξ that contains both y and z.*

Proof. Let ξ be an element of Ξ that contains y and z. We prove that ξ contains every point collinear to both y and z. Let w be such a point, and suppose for a contradiction that it is not contained in ξ. Then $T_y = \langle w, T_y(\xi)\rangle$ and $T_z = \langle w, T_z(\xi)\rangle$. In particular, the intersection $T_y \cap T_z$ is a plane. By Lemma 4.15, there is a point x with $\delta(x,z) = 1$ and $\delta(x,y) = 3$. The line xz lies in T_x, and must hence intersect $T_x \cap T_y$ in a point, a contradiction to Lemma 4.9. □

Proof of Theorem C. By the classification of 0-lacunary parapolar spaces [13, Table 2 p.11] X is isomorphic to $B_{3,3}(\star)$. Since both X and its dual are embeddable we obtain that X is isomorphic to $B_{3,3}(\mathbb{K}, \mathbb{A})$ by [30, Proposition 10.10].

We now show that $N = 7$. By assumption, there are two points x and y with $T_x \cap T_y = \emptyset$, thus $N \geq 7$. We prove that all points of X are contained in $Y = \langle T_x, T_y\rangle$.

Let $z \in X$ be such that $\delta(z,x) = 1$ and $\delta(z,y) = 2$, we prove that $T_z \subseteq Y$. Let ξ be the symp through z and y, then Y contains $T_y(\xi)$ and z, and hence ξ. Both the line xz and the plane $T_z(\xi)$ are contained in Y, hence $T_z \subseteq Y$.

Next, let z' be an arbitrary point on xz. There are at least two elements ξ_1, ξ_2 of Ξ that contain xz. By the previous argument, each of these two symps is contained in Y. Since $T_{z'} = \langle T_{z'}(\xi_1), T_{z'}(\xi_2)\rangle$, we find that $T_z' \subset Y$.

Since X is contained in the span of the above tangent spaces we obtain that $N = 7$.

By a result of Kasikova and Shult [19, p. 285] the absolute universal embedding exists and by Lemma 3.2 it occurs in dimension 7. □

5 Metasymplectic spaces

5.1 The axioms

Denote with $\mathbb{A}$ an associative division algebra. Suppose $N \in \mathbb{N} \cup \{\infty\}$, and denote with $\mathbb{P}^N(\mathbb{A})$ an N-dimensional projective space over $\mathbb{A}$. Let X be a spanning point set of $\mathbb{P}^N(\mathbb{A})$, let Ξ be a nonempty collection of polar spaces of rank 3, viewed as pairs $\xi = (\mathbb{P}(\xi), \mathscr{L}(\xi))$ with $\mathbb{P}(\xi)$ a 5-dimensional subspace of $\mathbb{P}^N(\mathbb{A})$ and $\mathscr{L}(\xi)$ the line set of the polar space ξ of rank 3 with thick hyperbolic lines that is fully embedded in $\mathbb{P}(\xi)$ and has point set $\mathbb{P}(\xi) \cap X$. Let Π be a (possibly empty) collection of planes such that, for all $\pi \in \Pi$, $\pi \cap X$ is a pair of distinct lines, intersecting in a point x_π. For $x \in X$ and $\xi \in \Xi$ with $x \in \mathbb{P}(\xi)$, we denote with $T_x(\xi)$ the tangent space at x to ξ, and let T_x be the subspace of $\mathbb{P}^N(\mathbb{A})$ generated by all these subspaces $T_x(\xi)$ for $x \in \xi \in \Xi$. We call every line of $\mathbb{P}^N(\mathbb{A})$ that either is a member of some $\mathscr{L}(\xi)$, $\xi \in \Xi$, or is contained in $X \cap \pi$, for some $\pi \in \Pi$ a *singular line* or *line* of X for short, if no confusion is possible. We again assume *connectivity*, that is, the graph on the points of X, adjacent when contained in a common member of Ξ, is connected. We also assume that for each point $p \in X$, the graph on the singular lines through p, adjacent if contained as members in a common $\mathscr{L}(\xi)$, for some $\xi \in \Xi$, is connected. We refer to the latter assumption as *local connectivity*. We impose the following axioms.

(F1) Every pair of intersecting singular lines of X is contained in some member of $\Xi \cup \Pi$ (with "contained in a member ξ of Ξ" we mean that the singular lines belong to $\mathscr{L}(\xi)$).

(F1′) For two points $x, y \in X$ we have $|T_x \cap T_y| > 1$ if and only if x, y are contained in a member of Ξ; if $|T_x \cap T_y| = 1$ then $T_x \cap T_y \subseteq X$ and the unique point of $T_x \cap T_y$ is collinear to both x and y (we say that the pair $\{x, y\}$ is *special*); finally there are points x, y with $T_x \cap T_y = \emptyset$.

(F2) For $\xi_1 \neq \xi_2 \in \Xi$, $\mathbb{P}(\xi_1) \cap \mathbb{P}(\xi_2) \subseteq X$.
(F3) For every point $x \in X$, the subspace T_x is at most 8-dimensional.

So we again do not assume that symps are convexly closed here.

We call (X, Ξ, Π) an *abstract metasymplectic variety* (AMV) and we will prove Theorem B in this section.

Terminology and Notation: The members of Ξ are called *symps*. If x and y are special points with $T_x \cap T_y = \{z\}$, we denote $z = x \bowtie y$. A point $x \in X$ is called *a bowtie* when $x = x_1 \bowtie x_2$ for some points $x_1, x_2 \in X$. A *hyperbolic line of* X is a hyperbolic line in some element $\xi \in \Xi$. Only if ξ is a symplectic polar space the corresponding hyperbolic line is a full line of $\mathbb{P}^N(\mathbb{A})$. Two points of X are *collinear* when they are contained in a singular line of X. A priori, a line L of $\mathbb{P}^N(\mathbb{A})$ can both be a line of X and a hyperbolic line of X. A *singular space* of X is a subspace of $\mathbb{P}(\mathbb{A})$ contained in X such that every pair of points of it is collinear in X. Two points that are not collinear but are contained in a symp are called *symplectic*.

We note that for each member $\xi \in \Xi$, the associated embedded polar space $(\mathbb{P}(\xi) \cap X, \mathscr{L}(\xi))$ has the property that no line of $\mathbb{P}(\xi)$ intersects X in exactly two points. Hence, if x and y are symplectic points, then the line $\langle x, y \rangle$ intersects X in at least three points. It follows that two symplectic points can never be contained in a common plane $\pi \in \Pi$.

Lemma 5.1. *Every singular line is contained in a symp.*

Proof. Let L be any singular line and pick a point $p \in L$. By connectivity, p is contained in a symp, hence in at least two singular lines. By local connectivity, L is contained in a symp. □

Lemma 5.2. *If x is collinear to distinct points y and z, which are either collinear or symplectic, then all points of the projective line $L = \langle y, z \rangle$ that belong to X are collinear to x.*

Proof. By Axiom (F1) there is a symp ξ containing xy and xz. Hence each point of L that belongs to X belongs to ξ. The lemma follows. □

Lemma 5.3. *If three nonconcurrent lines in a plane π of $\mathbb{P}(\mathbb{A})$ are lines of X, then π is a singular plane of X.*

Proof. By Lemma 5.2 all lines through the intersection points of the three lines are singular. Let L be an arbitrary line which intersects one of the lines in a point p. By the above p is contained in two singular lines, hence again by Lemma 5.2, L is singular. □

5.2 All symps are symplectic or every symp is convexly closed

Lemma 5.4. *Every symp $\xi \in \Xi$ that is not symplectic, is convexly closed.*

Proof. Let x, y be two noncollinear points of ξ, and suppose for a contradiction that $p \notin \xi$ is a point of X collinear to both x and y. Since hyperbolic lines are thick, it follows from (F1) that there is a symp $\xi' \in \Xi$ that contains the lines px and py. Note that $xy \subseteq \mathbb{P}(\xi) \cap \mathbb{P}(\xi')$. However, since ξ is not symplectic, the line xy contains points that are not in X, a contradiction to (F2). □

Lemma 5.5. *Let π be a plane of $\mathbb{P}(\mathbb{A})$ such that every pair of points of $\pi \cap X$ is contained in a symp. If $\pi \cap X$ contains a line, and a point outside of that line, then either $\pi \subset X$, or there is a symp $\xi \in \Xi$ with $\pi \subset \mathbb{P}(\xi)$.*

Proof. Suppose that $\pi \cap X$ contains a point p and a line L, and suppose that π contains a point q that does not belong to X. Denote with $\xi \in \Xi$ a symp that contains p and $L \cap pq$. Let q' be a point of $L \setminus pq$, and denote with $\xi' \in \Xi$ a symp that contains p and q'. By assumption, the polar space $(\mathbb{P}(\xi') \cap X, \mathscr{L}(\xi'))$ has thick hyperbolic lines, so there is a point x on $pq' \setminus \{p, q'\}$ that belongs to X. Denote with $\xi_x \in \Xi$ a symp that contains x and $qx \cap L$. Since $q \in \mathbb{P}(\xi_x) \cap \mathbb{P}(\xi)$, Axiom (F2) implies that $\xi = \xi_x$, implying that $\mathbb{P}(\xi)$ contains π. □

Lemma 5.6. *Either all symps of Ξ are symplectic, or no symp of Ξ is symplectic. In the latter case, all symps are convexly closed.*

Proof. Suppose that Ξ contains a symplectic symp ξ_1. Suppose that ξ_2 is a symp of Ξ such that $\pi = \mathbb{P}(\xi_1) \cap \mathbb{P}(\xi_2)$ has dimension at least two, and suppose for a contradiction that ξ_2 is not symplectic. It follows from (F2) that $\pi \subset X$. Since ξ_2 is not symplectic, every line in $\mathbb{P}(\xi_2)$ is a line of ξ_2, so π is a singular subspace of ξ_2, and hence a singular plane of X. Let $p \in \pi$ be such that $\pi \subset T_p(\xi_1)$. Let $q \in \pi$ be different from p, and select $r \in \pi$ not on pq. Moreover, for $i = 1, 2$, let x_i be a point of X on $T_p(\xi_i) \cap T_q(\xi_i) \cap X$, not in π. Both x_1 and r are points of ξ_1, so the line $x_1 r$ is contained in X. Since q is collinear to r (inside ξ_2 and to x_1 (inside ξ_1), Lemma 5.2 implies that q is collinear to each point of rx_1. Since each point of rx_1 is also collinear to p (inside ξ_1), Lemma 5.2 implies that each point of rx_1 is collinear to pq. In ξ_2, each point of $rx_2 \cap X$ is collinear to each point of pq. Now let x be an arbitrary point of the plane α spanned by r, x_1, x_2 not on rx_1 and distinct from x_2. Set $y = rx_1 \cap xx_2$. Then by the previous arguments, y and x_2 are collinear or symplectic (both are collinear to p and q), and so Lemma 5.2 again implies that p and q are collinear to x. It follows that all points of $\alpha \cap X$ are collinear to both p and q and hence every pair of such points is symplectic. Since x_2 and r are noncollinear points of ξ_2, there are points on the line rx_2 that are not in X. It then follows from Lemma 5.5 that there is a symp that contains α. But, by Lemma 5.4, the symp ξ_2 is the unique symp that contains x_2 and r. This is a contradiction to $x_1 \notin \pi$.

Next, suppose that ξ_2 is a symp of Ξ such that $L = \mathbb{P}(\xi_1) \cap \mathbb{P}(\xi_2)$ has dimension one, and suppose again for a contradiction that ξ_2 is not symplectic. Let p and q be distinct points of L, let x_1 be a point of $T_p(\xi_1) \cap T_q(\xi_1)$ not on L, and let x_2, y_2 be noncollinear points in X contained in $T_p(\xi_2) \cap T_q(\xi_2)$. First suppose that every point of $x_1 x_2$ is contained in X. Since x_1 and x_2 are collinear to both p and q, they are symplectic, and so Lemma 5.2 implies that each point of $x_1 x_2$ is collinear to both p, q. Since also y_2 is collinear to both p and q, the argument above with α can be copied for the plane β spanned by x_1, x_2, y_2 showing that all points of $\beta \cap X$ are collinear to both p and q, and so each pair of points of $\beta \cap X$ is symplectic. Then it follows from Lemma 5.5 and the fact that $x_2 y_2$ contains points not in X, that there is a symp $\xi \in \Xi$ that contains β. This symp ξ is not symplectic as β is not contained in X. It follows from Lemma 5.4 that $\xi = \xi_2$, and hence that $x_1 \in \xi_2$, a contradiction. We hence obtain that there are points on $x_1 x_2$ that are not in X. Let $\xi' \in \Xi$ be a symp that contains x_1 and x_2. This symp is not symplectic, so Lemma 5.4 implies that it is convexly closed, and in particular, that it contains p and q. We hence obtain that $\mathbb{P}(\xi') \cap \mathbb{P}(\xi_1)$ and $\mathbb{P}(\xi') \cap \mathbb{P}(\xi_2)$ both have dimension at least two, the contradiction then follows from the previous paragraph.

Now the assertion follows from the connectivity and the local connectivity. □

We now introduce the notion of the residue at a point and show in the next subsection that, under a mild condition, it is an abstract dual polar variety.

Definition 5.7. Let $x \in X$ be arbitrary and let C_x be a subspace of T_x of co-dimension 1 not containing x. Let X_x be the set of points of C_x which are contained in a singular line of X

with x. For each $\xi \in \Xi$ containing x, let ξ_x be the generalized quadrangle obtained from ξ by intersecting C_x with each member of $\mathscr{L}(\xi)$ that contains x. Define $\mathbb{P}(\xi_x) = T_x(\xi) \cap C_x$ and $\mathscr{L}(\xi_x) = \{L \in \mathscr{L}(\xi) \mid L \subseteq C_x \cap T_x(\xi)\}$. We view ξ_x as the pair $(\mathbb{P}(\xi_x), \mathscr{L}(\xi_x))$. Let Ξ_x be the collection of all such pairs ξ_x for $x \in \xi$. Then we call (X_x, Ξ_x) the *residue of* (X, Ξ, Π) *at* x and denote it by $\mathsf{Res}(x)$.

5.3 All points of a diameter 3 residue are polar

In this subsection **all notation and terminology refer to elements of** $\mathsf{Res}(x)$ **for a fixed point** $x \in X$ unless explicitly mentioned otherwise.

For every point y in $\mathsf{Res}(x)$, we denote

$$T_y^x := \langle \pi \mid \pi \text{ is singular plane of } X \text{ through } xy \rangle \cap C_x$$

Two points y and z of $\mathsf{Res}(x)$ are collinear in $\mathsf{Res}(x)$ if and only if they are collinear in X. Moreover $d(y,z)$ denotes the distance in $\mathsf{Res}(x)$ between points y and z of $\mathsf{Res}(x)$. Given a point y of $\mathsf{Res}(x)$ not contained in a given symp ξ of $\mathsf{Res}(x)$, we call a point z of ξ the *gate* of ξ for y if z is the unique point of ξ collinear with y.

Lemma 5.8. *The point-line space* $\mathsf{Res}(x)$ *is connected, in other words,* $d(y,z) \in \mathbb{N}$ *for all points* y *and* z *of* $\mathsf{Res}(x)$.

Proof. This follows by local connectivity and Axiom (F1). □

Lemma 5.9. *For two points* y *and* z *of* $\mathsf{Res}(x)$, *we have* $d(y,z) \geq 3$ *if, and only if,* $x = y \bowtie z$.

Lemma 5.10. *For* $\xi = (\mathbb{P}(\xi), \mathscr{L}(\xi)) \in \Xi_x$, *one has that* ξ *is a generalized quadrangle, with* $\dim(\mathbb{P}(\xi)) = 3$. *For a point* y *of* ξ, *the subspace* $T_y^x(\xi)$ *is a plane.*

Lemma 5.11. *Every pair of points of* $\mathsf{Res}(x)$ *at distance at most two (in* $\mathsf{Res}(x)$*) is contained in a common symp of* $\mathsf{Res}(x)$, *which is not necessarily unique.*

Proof. Let y, z with $d(y,z) \leq 2$. Then $T_y^x \cap T_z^x$ contains some point p, implying that both x and p are contained in $T_y \cap T_z$. There is hence a symp in Ξ that contains xy and xz by Axiom (F1) which yields a member of Ξ_x containing y and z. □

While it follows from Lemma 5.11 that every two collinear points y and z of $\mathsf{Res}(x)$ are contained in a symp $\xi \in \Xi_x$, it is a priori not clear whether y and z are collinear in ξ.

Lemma 5.12. *For every point* y *of* $\mathsf{Res}(x)$, *we have* $\langle T_y^x(\xi) \mid \xi \in \Xi_x, y \in \xi \rangle \subseteq T_x \cap T_y \cap C_x = T_y^x$.

Definition 5.13. A point y of $\mathsf{Res}(x)$ is called a *polar point* when there is some point z of $\mathsf{Res}(x)$ for which $x = y \bowtie z$.

Lemma 5.14. *Suppose that* y *is a polar point of* $\mathsf{Res}(x)$. *Then* $\dim(T_y^x) = 3$ *and moreover* $\langle T_y^x(\xi) \mid \xi \in \Xi_x, y \in \xi \rangle = T_y^x$.

Proof. Using Lemma 5.11, we find a symp $\xi \in \Xi_x$ that contains y. Since $T_y^x(\xi) \subseteq T_y^x$, we find $\dim(T_y^x) \geq \dim(T_y^x(\xi)) = 2$. Suppose for a contradiction that $\dim(T_y^x) = 2$. Since y is a polar point, Lemma 5.9 implies that there is some point z with $d(z,y) \geq 3$. Without loss of generality, we may assume that $d(y,z) = 3$. Let w be a point collinear to y for which $d(w,z) = 2$. By

Lemma 5.11, there exists a symp $\xi' \in \Xi_x$ that contains w and z. Let w_1 and w_2 be two distinct points of ξ' collinear (in ξ') to w and z. For $i = 1, 2$, we have $d(y, w_i) \leq 2$, so, again by Lemma 5.11, there is a symp $\xi_i \in \Xi_x$ through y and w_i. Note that $T^x_y(\xi_i)$ is 2-dimensional, and hence coincides with T^x_y. In particular, this implies that $w \in T^x_y \subseteq \xi_i$, and hence that $ww_i \subseteq \xi_i$.

We claim that for every point p of $T^x_y \setminus yw$, the point p is collinear to a (hyperbolic) line of $T^x_w(\xi')$. Let p be such a point. In ξ_i, the point p is collinear to some point w'_i of ww_i. Since $(\mathbb{P}(\xi_i), \mathscr{L}(\xi_i))$ is a generalized quadrangle, $w'_i \neq w$. By Lemma 5.2, the point p is indeed collinear to the (hyperbolic) line $h_p = X \cap w_1w_2$.

Let p be a point of $T^x_y \setminus yw$ as in the previous paragraph, and let p' be a point on $py \setminus \{p, y\}$. By the foregoing, both p and p' are collinear to a hyperbolic line h_p and $h_{p'}$, respectively, of $T^x_w(\xi')$. Since $T^x_w(\xi')$ is a plane, there is a point $q \in \langle h_p \rangle \cap \langle h_{p'} \rangle$.

If $q \in X$ then q is collinear to p and p', so, by Lemma 5.2, also to y. Note that y is collinear to both q and w, so after again applying Lemma 5.2, we obtain $wq \subseteq T_y$. The line wq however intersects $w_1w_2 \subseteq T_z$ in a point, so this implies that $T_y \cap T_z \neq \emptyset$, a contradiction.

If $q \notin X$, then ξ' is not symplectic, and is, by Lemma 5.4, convexly closed. Both p and p' are collinear to two noncollinear points of ξ', which implies that $y \in pp' \subset \xi'$, a contradiction to $d(y, z) = 3$. We conclude that $\dim(T^x_y) \geq 3$. The same arguments applied to z, yield $\dim(T^x_z) \geq 3$. The assertion then follows from the fact that $T^x_y \cap T^x_z = \emptyset$. □

Corollary 5.15. *If y is a point of* $\mathsf{Res}(x)$ *with* $\dim(T^x_y) = 3$, *and ξ_1, ξ_2 are two symps of* $\mathsf{Res}(x)$ *that contain y, then there is a line through y all of whose points belong to* $\mathsf{Res}(X)$ *that is contained in both* $\mathscr{L}(\xi_1)$ *and* $\mathscr{L}(\xi_2)$. *This holds in particular when y is a polar point.* □

Lemma 5.16. *In* $\mathsf{Res}(x)$, *the points at distance at most* 2 *from a given point form a subspace.*

Proof. Let y be a point of $\mathsf{Res}(x)$. If $d(y, z) \leq 2$ for all points z of $\mathsf{Res}(x)$, there is nothing to prove. Suppose therefore that there is a point z of $\mathsf{Res}(x)$ with $d(y, z) = 3$. Note that both y and z are polar points. Suppose for a contradiction that we find a line L through z in $\mathsf{Res}(x)$ that contains two points z_1 and z_2 with $d(y, z_1) = d(y, z_2) = 2$. Using Lemma 5.11, we find symps ξ_1 and ξ_2 (which are distinct as z does not belong to either) of Ξ_x that contain y, z_1 and y, z_2, respectively. By Corollary 5.15, there is a line K through y contained in both ξ_1 and ξ_2. For $i = 1, 2$, denote with w_i the point on K collinear with z_i in ξ_i. The symps ξ_1 and ξ_2 are symplectic because otherwise they coincide with the symp through w_1 and z_2, and with the one through w_2 and z_1, and then it also contains z, a contradiction.

Then w_1z_2 and w_2z_1 are hyperbolic lines in ξ_2 and ξ_1, respectively. Since z_1 is collinear to both z_2 and w_1, we conclude by Lemma 5.2 that z_1 is collinear to every point of w_1z_2. Likewise, z_2 is collinear to every point of w_2z_1, w_1 is collinear to each point of w_2z_1 and w_2 is collinear to each point of w_1z_2. Another application of Lemma 5.2 shows now that each point of w_1z_2 is collinear to each point of w_2z_1. Hence every point on the line yz is on a singular line of X, contradicting the fact that the plane $\langle x, y, z \rangle$ intersects X in $xy \cup xz$. This final contradiction proves the assertion. □

This has the following immediate consequence.

Corollary 5.17. *Let ξ be a symp in* $\mathsf{Res}(x)$, *and let y be a point collinear with some point z of ξ but at distance* 3 *from another point of ξ, then z is the gate of ξ for y, and every point of ξ noncollinear to z in ξ is at distance* 3 *from y.* □

Lemma 5.18. *Let ξ be a symp of* $\mathsf{Res}(x)$*, and let y be a point of* $\mathsf{Res}(x)$*, and assume that the point z of* $\mathsf{Res}(x)$ *is the gate of ξ for y. If z' is a point of ξ with* $\dim(T^x_{z'}) = 3$ *and* $d(y, z') = 2$*, then every symp through y and z' contains z.*

Proof. Let ξ' be any symp of $\mathsf{Res}(x)$ through y and z'. Using Corollary 5.15 we see that $\xi \cap \xi'$ contains a line $M \ni z'$. The point y is collinear to a point of M (inside ξ'), but z is the unique point of ξ collinear to y, which implies that $z \in M \subseteq \xi'$. □

Lemma 5.19. *Let ξ be a symp of* $\mathsf{Res}(x)$*, and let y be a point of* $\mathsf{Res}(x)$*. Assume that y is collinear to some point of ξ, and is at distance* 3 *from some other point of ξ. If there is some point z' of ξ with* $d(y, z') = 2$ *and* $\dim(T^x_{z'}) = 3$*, then every point of ξ is polar.*

Proof. Let z be a point of ξ collinear to y. By Corollary 5.17, the point z is the gate in ξ for y. Let y' be a point of ξ. If y' is noncollinear to z in ξ, then, again by Corollary 5.17, we find $d(y, y') = 3$, which implies that y' is polar. Suppose that y' is collinear to z, but is not on the line zz'. Let ξ' be a symp through y and z'. By Lemma 5.18, this symp contains the line zz'. Let w' be a point of ξ' not collinear to z or z', and let w be a point of ξ collinear to z' but not to z. Note that $d(y, w) = 3$, so by Corollary 5.17, the point w is at distance three from every point in ξ' noncollinear to z', and in particular to w'. Applying Corollary 5.17 to w' and ξ we obtain $\delta(w', y') = 3$, which implies that y' is polar, and in particular, that $\dim(T^x_y) = 3$. By switching the roles of z' and y', we also find that every point of $zz' \setminus \{z\}$ is polar. The point w' can however play the same role as y, so we also find that z is polar. □

We will usually apply this lemma in the following, weaker, form:

Corollary 5.20. *Let y be a polar point of* $\mathsf{Res}(x)$ *which is collinear to a unique point of the symp ξ, which contains a point at distance* 3 *from y. Then all points of ξ are polar as soon as some point of ξ at distance* 2 *from y is polar.*

Lemma 5.21. *Let ξ be a symp in* $\mathsf{Res}(x)$*. If there is some point y in ξ for which $\mathbb{P}(\xi) \subseteq T^x_y$, then $\mathbb{P}(\xi) \subseteq T^x_z$ for all points z of ξ.*

Proof. Let w be any point of ξ noncollinear to y in ξ. Then wy is a line of X which is not a line of ξ, implying that $\mathbb{P}(\xi) \subseteq T^x_w$. Let v then be an arbitrary point of ξ. For every point w of ξ noncollinear to z in ξ, we find $wv \subseteq T^x_v$. In particular, this implies that $\mathbb{P}(\xi) \subseteq T^x_v$. □

Lemma 5.22. *Let ξ be a symp of* $\mathsf{Res}(x)$ *containing a polar point y. Then $T^x_y \cap \mathbb{P}(\xi) = T^x_y(\xi)$.*

Proof. Suppose for a contradiction that there is some point z of ξ for which $\mathbb{P}(\xi) \subseteq T^x_z$. By Lemma 5.21, this is the case for all points z of ξ. Since $\mathsf{Res}(x)$ contains a polar point, it has diameter at least three by Definition 5.13 and so we find some point w not contained in ξ. Without loss of generality, we may assume that w is collinear to some point v of ξ. Note that $\mathbb{P}(\xi) \subseteq T^x_v$ implies that v is collinear to y, and hence that $d(w, y) = 2$. We can hence consider a symp ξ' containing w and y. Since $\dim(T^x_y) = 3$ and since T^x_y contains $\mathbb{P}(\xi)$, we obtain $T^x_y = \mathbb{P}(\xi)$, and hence $T^x_y(\xi') \subseteq \mathbb{P}(\xi)$. By Lemma 5.16, there is at least one other polar point y' in $T^x_y(\xi)$, hence for which $\dim(T^x_{y'}) = 3$, and hence for which $T^x_{y'}(\xi') \subseteq \mathbb{P}(\xi)$. We however have that $\mathbb{P}(\xi') = \langle T^x_y(\xi'), T^x_{y'}(\xi') \rangle = \mathbb{P}(\xi)$, which implies that $w \in \xi$, a contradiction. □

Lemma 5.23. *Let y be a polar point of* $\mathsf{Res}(x)$*, and let $y' \perp y$. If y' is not a polar point, then there exists a singular plane containing the line yy'.*

Proof. Since y is polar, there exists a point z with $d(y,z)=3$. If y' is not polar, then $d(y',z)=2$. Let ξ be a symp containing z and y', and let w be a point in ξ collinear to both z and y'. Since $d(w,y)=2$, there is some symp ξ' containing w and y.

We claim that ξ' does not contain y'. Suppose for a contradiction that this would be the case, and take v in ξ' collinear to neither w nor y. Since w is the gate of ξ' for z, we find $d(v,z)=3$. Note that v is collinear to a point v' of wy', different from y' which is the gate of ξ for v. All points of ξ noncollinear to v' are hence polar. In particular there exists a polar point z' collinear to y' not on $y'v'$. Corollary 5.20 (applied to v, ξ and z') implies that all points of ξ are polar, a contradiction to the assumption that y' is not polar. We conclude that ξ' indeed does not contain y'.

Let v be a point of wy' different from w and y'. Since $d(y,v)=2$, there exists a symp ξ'' through y and v. Note that $\xi'' \cap wy' = \{v\}$, for if $wy' \subset \xi''$, then the symp ξ'' would contain both y, y' and w, which by the argument above, cannot be the case. The symps ξ' and ξ'' both contain y, so, since y is polar, there is a line L through y which is both a line of ξ' and of ξ''.

Denote with w' and v' the points on L collinear to w and v respectively. We claim that $w'=v'$. Suppose for a contradiction that $w' \neq v'$. Then, since w is the gate of ξ' for z, the point v' is polar. Let u be a point of $\xi \setminus wy'$ collinear to v. Since $d(u,y)=3$, the point v is the gate of ξ'' for u. Corollary 5.20 implies that all points of ξ'', in particular v, are polar, and applying Corollary 5.20 once more implies that all points of ξ, including y', are polar, a contradiction. Hence $w'=v'$, thus w' is collinear to both w and v of wy', and hence, by Lemma 5.2, also to y'. The lines yy', yw' and $y'w'$ are three pairwise concurrent lines not containing a common point. The assertion then follows from Lemma 5.3. $\square$

Lemma 5.24. *A polar point is contained in at most one singular plane.*

Proof. Let y be a polar point. Suppose for a contradiction that y is contained in two singular planes π_1 and π_2. Since $\dim(T^x_y)=3$, π_1 and π_2 intersect in a line L. By Lemma 5.16, we can find a polar point z on L different from y. Since T^x_z contains both π_1 and π_2, and since $\dim(T^x_z)=3$ we obtain $T^x_y = T^x_z$. Let ξ be a symp that contains both y and z. By Lemma 5.22 the subspace $T^x_y \cap \mathbb{P}(\xi)$, which equals $T^x_z \cap \mathbb{P}(\xi)$, is a plane π. But then $\pi = T^x_y(\xi) = T^x_z(\xi)$, a contradiction. $\square$

Lemma 5.25. *If* $\mathsf{Res}(x)$ *contains at least one polar point, that is, if x is a bowtie, then every point of* $\mathsf{Res}(x)$ *is polar.*

Proof. By assumption, we find points z and y with $d(z,y) \geq 3$, hence y and z are polar. By connectivity of $\mathsf{Res}(x)$, it suffices to prove that every point collinear to y is polar. Suppose for a contradiction that $y' \perp y$ is not polar. By Lemma 5.23, there is a singular plane π in T^x_y through yy', and by Lemmas 5.23 and 5.24, all points of T^x_y not on π are polar. Since y' is not polar, we have $d(z,y')=2$. Let ξ be a symp containing z and y', and let w be a point of ξ collinear to both z and y'. Since $d(w,y) \leq 2$, there exists a symp ξ' containing w and y. We claim that this symp ξ' intersects T^x_y in a plane distinct from π.

Indeed, if not, then by Lemma 5.22, we have $T^x_y(\xi') = \pi$. Hence $T^x_u(\xi') \neq \pi$ for each point $u \in yy' \setminus \{y, y'\}$ (use Lemma 5.16 and Lemma 5.22), and so $T^x_u = \langle T^x_u(\xi'), \pi \rangle = \mathbb{P}(\xi')$, contradicting Lemma 5.22. The claim follows.

Now the point w is the gate of ξ' for z, and every point of $T^x_y(\xi')$ not in π is polar. By Lemma 5.19 all points of ξ', and in particular w, are polar. Applying Lemma 5.19 again to ξ, we find that all points of ξ are polar, and in particular that y' is polar after all. $\square$

Corollary 5.26. *If x is a bowtie, then* $\mathsf{Res}(x)$ *is an abstract dual polar variety, and as such* $(X, \mathscr{L}) \cong \mathsf{B}_{3,3}(\mathbb{K}, \mathbb{A})$, *for some field* $\mathbb{K}$ *over which* $\mathbb{A}$ *is an associative quadratic division algebra.*

Proof. Since for each point y of $\mathsf{Res}(x)$, the tangent space T_y^x is 3-dimensional, all axioms of a dual polar variety are satisfied. The last assertion then follows from Theorem C. □

In particular, we will be using the following properties of dual polar spaces of rank 3:

Corollary 5.27. *Suppose that $x \in X$ is a bowtie.*

(1) *Every point collinear to x is collinear to a line of every symp through x.*
(2) *Every two symps through x that intersect in a line, intersect in a plane.*
(3) *Every point collinear to x is symplectic to at least one point of each line collinear to x' but not containing x'.*

We end this subsection with a sufficient condition for the point x to be a bowtie.

Lemma 5.28. *Suppose that* $\mathsf{Res}(x)$ *contains symps ξ_1 and ξ_2 for which $\mathbb{P}(\xi_1) \neq \mathbb{P}(\xi_2)$ and every point y of $\xi_1 \cup \xi_2$ satisfies* $\dim(T_y^x) \leq 3$. *Then x is a bowtie.*

Proof. Suppose for a contradiction that x is not a bowtie. Then, by definition, the diameter of $\mathsf{Res}(x)$ is equal to 2.

First assume that $\mathbb{P}(\xi_1) \cap \mathbb{P}(\xi_2) = \emptyset$. Select a point $x_1 \in \xi_1$ and $x_2 \in \xi_2$. Since the diameter is 2, there is a symp ζ of $\mathsf{Res}(x)$ containing x_1 and x_2. By the assumption on the tangent spaces, ζ and ξ_i have at least one line L_i through x_i in common, $i = 1, 2$. We may rename x_2 on L_2 so that $x_1 \perp x_2$. Now pick x_2' in ξ_2 not collinear to x_2 in ξ_2. The symp pjζ' through x_1 and x_2' also has a line L_2' through x_2' in common with ξ_2 and since $x_2 \notin L_2'$, this yields a second point of ξ_2 collinear to x_1, implying that $T_{x_1}^x$ is at least 4-dimensional (as it intersects ξ_1 in at least a plane and ξ_2 in at least a (disjoint) line). This contradiction shows that we may assume that there exists some point $z \in \xi_1 \cap \xi_2$. Since $\dim(T_z^x) \leq 3$, we find that $T_z(\xi_1) \cap T_z(\xi_2)$ contains at least a line, implying that $\xi_1 \cap \xi_2$ is at least a line, and hence that $\dim(\langle \xi_1, \xi_2 \rangle) \leq 5$.

We claim that all points of $\mathsf{Res}(x)$ are contained in $V := \langle \mathbb{P}(\xi_1), \mathbb{P}(\xi_2) \rangle$. Suppose first that $\mathbb{P}(\xi_1) \cap \mathbb{P}(\xi_2)$ is a plane π. Let p_i be the point of π such that $T_{p_i}^x(\xi_i) = \pi$. If $p_1 \neq p_2$, then $T_{p_1}^x \cap \mathbb{P}(\xi_1) = \pi$ as there are points of ξ_2 not in $\mathbb{P}(\xi_1)$ collinear to p_1. On the other hand, the same argument shows that $T_{p_2}^x = \mathbb{P}(\xi_1)$. This contradicts Lemma 5.21.

Consequently $p_1 = p_2 =: p$. Let y be any point of $\mathsf{Res}(x) \setminus \pi$ and let ζ be a symp through p and y (which exists as the diameter of $\mathsf{Res}(x)$ is equal to 2). Then $T_p^x(\zeta)$ intersects $\pi = T_p^x(\xi_1) = T_p^x(\xi_2)$ in at least a line M. Take an arbitrary point $u \in M \setminus \{p\}$. Then by assumption $\dim T_u^x \leq 3$, hence $T_u^x = \langle T_u^x(\xi_1), T_u^x(\xi_2) \rangle \subseteq V$. This implies $T_u^x(\zeta) \subseteq V$. Hence, if $u' \in M \setminus \{p, u\}$, then $y \in \mathbb{P}(\zeta) = \langle T_u^x(\zeta), T_{u'}^x(\zeta) \rangle \subseteq V$ and the claim follows. Note that, with the previous notation, $T_u^x \neq T_{u'}^x$, as otherwise both equal $\mathbb{P}(\xi_1)$ and $\mathbb{P}(\xi_2)$. Hence for at least one of both, say u, holds $T_u^x \neq \mathbb{P}(\zeta)$. Then Lemma 5.21 implies that T_y^x does not contain $\mathbb{P}(\zeta)$ and so, since $\dim V = 4$, we conclude $\dim T_y^x \leq 3$.

Now assume that $\mathbb{P}(\xi_1) \cap \mathbb{P}(\xi_2)$ is a line L. Note that, for each point $y \in L$, the assumption $\dim T_y^x \leq 3$ implies that $T_y^x(\xi_1) \cap T_y^x(\xi_2) \subseteq \mathbb{P}(\xi_1) \cap \mathbb{P}(\xi_2) = L$. This implies that $L \in \mathscr{L}(\xi_1) \cap \mathscr{L}(\xi_2)$.

(*i*) For each point y on L, the tangent space T_y^x contains points of ξ_2, hence T_y^x is not contained in $\mathbb{P}(\xi_1)$. By Lemma 5.21, this implies, for each $z \in \xi_1$, that $T_z^x \cap \mathbb{P}(\xi_1) = T_z^x(\xi_1)$. Similarly for the points of ξ_2.

(*ii*) Let $z_1 \in \xi_1 \setminus L$ and pick $z_2 \in \xi_2$ such that z_2 is not collinear to $z_1^{\perp} \cap L$. As the diameter of $\mathsf{Res}(x)$ is equal to 2, there is a symp ζ that contains both z_1 and z_2. This symp intersects ξ_2 in a line L_2 (remember by assumption $\dim T^x_{z_2} \leq 3$). The point z_1 is collinear with a point w of this line, and by construction $w \notin \xi_1$. This implies that $T^x_{z_1} = \langle w, T^x_{z_1}(\xi_1) \rangle \subseteq V$. Since z_1 was arbitrary, we see that every point of $\mathsf{Res}(x)$ collinear to some point of ξ_1 belongs to V.

(*iii*) Let z be any point, and by (*ii*) we may suppose it is not collinear to any point of ξ_1. Since the diameter of $\mathsf{Res}(x)$ is 2, it is contained in a symp ζ together with some arbitrary chosen point $z_1 \in \xi_1$. Then ζ intersects ξ_1 in a line (as $T^x_{z_1}$ has dimension 3), so, looking inside ζ, the point z is collinear to some point of ξ_1, a contradiction. Hence z is contained in V, and the claim is proved

Now, still assuming that $\mathbb{P}(\xi_1) \cap \mathbb{P}(\xi_2) = L$, suppose for a contradiction that there exists a point y with $\dim T^x_y \geq 4$. Then, by dimension, y is collinear to all points of a plane π of ξ_1. Let p be the point of π for which $T^x_p(\xi_1) = \pi$. Note that $T^x_p = \langle \pi, y \rangle \subseteq T^x_y$. Let ζ be a symplecton that contains both p and y. Since $\dim T^x_p \leq 3$, the symplecton ζ shares a singular line $K \subseteq T^x_p(\xi_1) \cap T^x_p(\zeta)$ through p with ξ_1. Then T^x_y contains K. But since K clearly does not belong to $T^x_y(\zeta)$, we deduce $\mathbb{P}(\zeta) \subseteq T^x_y$. Hence, using Lemma 5.21, we find that all points of ζ have ζ in their tangent space. But then for each point z of K we have $T^x_z = \mathbb{P}(\zeta)$, which would imply that $T^x_z(\xi_1)$ is independent of $z \in K$, a contradiction.

This now implies that in this case no pair of symps intersects in a plane, as these symps satisfy the assumptions of the lemma and hence the first part of the proof would imply $\dim V = 4$. Hence, if ξ_1 and ξ_2 intersect in a line, then the intersection of any pair of symps is a line, and the same argument that showed that L is singular shows that the intersecting line of two symps is always a singular line. We conclude that $\mathsf{Res}(x)$ defines a convexly closed geometry, hence a parapolar space. By Main Result 1.1 of [14], $\mathsf{Res}(x)$ has nonthick symps, a contradiction since we have thick hyperbolic lines in each symp.

Hence we may assume that $\mathbb{P}(\xi_1) \cap \mathbb{P}(\xi_2)$ is a plane π, and $\dim V = 4$. If for each point $u \in (\mathbb{P}(\xi_1) \cup \mathbb{P}(\xi_2)) \setminus \pi$ the dimension of T^x_u is exactly equal to 2, then the symp through points $u_1 \in \mathbb{P}(\xi_1) \setminus \pi$ and $u_2 \in \mathbb{P}(\xi_2) \setminus \pi$ with $T^x_{u_1} \cap \pi \neq T^x_{u_2} \cap \pi$ would generate V, a contradiction. Hence there exists a point u in ξ_1 not in π with $\dim T^x_u = 3$. Pick a line $L_1 \in \mathscr{L}(\xi_1)$ through u and a disjoint line $L_2 \in \mathscr{L}(\xi_1)$ contained in π. Then we find distinct points $y_i, z_i \in L_i$, $i = 1, 2$ such that the tangent spaces at y_i and z_i are 3-dimensional (and distinct). Since V is 4-dimensional, the intersection $T^x_{y_i} \cap T^x_{z_i}$ is a plane α_i, which, by Lemma 5.3, is a singular plane intersecting $\mathbb{P}(\xi_1)$ in L_i, for $i = 1, 2$. It follows that $\alpha_1 \cap \alpha_2$ is a point u. Since $\alpha_1 \cup \alpha_2$ generates V, we deduce $T^x_u = V$, contradicting the conclusion of the third paragraph of this proof.

This final contradiction shows the assertion. □

5.4 All points of X are bowties

A 3-*path* is a tuple (x, y, z, w) such that $x \perp y \perp z \perp w$ and $\delta(x, w) = 3$.

Lemma 5.29. *For points $x, y \in X$, we have $T_x \cap T_y = \emptyset \iff \delta(x, y) \geq 3$. Moreover, there exist points $x_1, x_2 \in X$ with $T_{x_1} \cap T_{x_2} = \emptyset$ and $\delta(x_1, x_2) = 3$.*

Proof. The first claim is immediate. By (F1′), there exist points x, y with $T_x \cap T_y = \emptyset$, and hence $\delta(x, y) \geq 3$, in particular, there are two points x_1, x_2 with $\delta(x_1, x_2) = 3$. □

5.4.1 When there is a bowtie on a 3-path

Lemma 5.30. *Let x_1 and x_2 be points in X with $\delta(x_1, x_2) = 3$, and let (x_1, x, x', x_2) be a 3-path. If x is a bowtie, then no symp contains both x and x_2. In particular, x' is a bowtie. Moreover, $x = x_1 \bowtie x'$ and $x' = x \bowtie x_1$.*

Proof. Suppose that x and x_2 are contained in a symp ξ. The point x is a bowtie, so by Corollary 5.27, the point x_1 is collinear to a line L of ξ. The point x_2 is contained in ξ, and is hence collinear to a point of L, a contradiction to $\delta(x_1, x_2) = 3$. We find that x and x_2 are special and hence $x \bowtie x_2 = x'$, thus x' is a bowtie. Reversing the roles of x_1 and x_2 yields $x = x_1 \bowtie x'$. ☐

For collinear points x, y, we denote, abusing notation, by T^x_y the tangent space in $\mathsf{Res}(x)$ at the point $C_x \cap xy$.

Lemma 5.31. *If the points x and y of X are collinear, then $\langle x, T^x_y \rangle = \langle y, T^y_x \rangle \subseteq T_x \cap T_y$. In particular, T^x_y and T^y_x have the same dimension.*

Proof. It is clear that $\langle x, T^x_y \rangle \subseteq T_x \cap T_y$. By definition, the subspace $\langle y, T^y_x \rangle$ is the subspace spanned by all planes of X through x and y, and hence coincides with $\langle x, T^x_y \rangle$. ☐

Lemma 5.32. *Let x_1 and x_2 be points in X with $\delta(x_1, x_2) = 3$, and let (x_1, x, x', x_2) be a 3-path. Then $T^{x_1}_x$ and $T^x_{x_1}$ have dimension at most 3.*

Proof. First suppose that x is a bowtie. Then it follows from Lemma 5.14 and Lemma 5.25 that $T^x_{x_1}$ has dimension 3, in which case the claim follows from Lemma 5.31. Suppose that x is not a bowtie. By Lemma 5.30, the point x' is not a bowtie either. So, in particular, we find a symp ξ containing xx' and $x'x_2$. Note that $T_x(\xi) \cap T_{x_2}(\xi)$ has dimension 3 and is by (F1) contained in $T_x \cap T_{x_2}$. By assumption, we have that $T_{x_1} \cap T_{x_2} = \emptyset$, so the fact that T_x has dimension at most 8 implies that $T_x \cap T_{x_1}$ has dimension at most 4. The claim now follows as $T^{x_1}_x$ and $T^x_{x_1}$ have dimension at most $\dim(T_x \cap T_{x_1}) - 1$. ☐

Lemma 5.33. *Let (x_1, x, x', x_2) be a 3-path with x and x' bowties. Then x_1 and x_2 are also bowties.*

Proof. Without loss of generality we prove this for x_1 and we may assume, for a contradiction, that x_1 is not a bowtie. Let L be an arbitrary line through x_1 distinct from xx_1 and let ξ be a symplecton containing L and xx_1 (which exists by Axiom (F1)). Since x is a bowtie, x' is collinear to a line M of ξ through x.

If there exists a point z on M which is not a bowtie, then let p be any point on L collinear in ξ to z (possibly $p = x_1$). Let ξ' be a symp containing p and x' (which exists since we assume that z is not a bowtie). Since x' is a bowtie, x_2 is collinear to a line N in ξ'. Let p' be a point in ξ' on N collinear to p. Then we have found a 3-path (x_2, p', p, x_1), which reduces to a 2-path if $p = x_1$. Hence $p \neq x_1$ and the 3-path contains a point p on L distinct from x_1.

Assume now that all points on M are bowties. Since x' is a bowtie, Corollary 5.27(3) yields a symp ξ'' containing x_2 and a point z on M, which is a bowtie. Hence, by Corollary 5.27(1), a point p on L collinear in ξ to z is itself collinear to a line K of ξ'' through z. Then, inside ξ'', we find a point $u \in K$ collinear to x_2. Hence we again obtain a 3-path (x_1, p, u, x_2), which again cannot reduce and contains a point p of L different from x_1. Then Lemma 5.32 yields $\dim T^{x_1}_p \leq 3$.

Since L was an arbitrary line through x_1, x_1 is a bowtie after all by Lemma 5.28. ☐

A *bowtie path* is a sequence (x_1, x, x', x_2) of bowtie points of X such that $x = x_1 \bowtie x'$ and $x' = x_2 \bowtie x$. An example is a 3-path of bowtie points. In Lemma 5.37 we will show the converse.

Lemma 5.34. *Let (x_1, x, x', x_2) be a bowtie path and let $y, y' \in X$ be such that $x_1 \perp y \perp y' \perp x_2$, $y \perp x$ and $y' \perp x'$. If x, x', y, y' are contained in a common symp ζ, then both y and y' are bowties, $y = x_1 \bowtie y'$ and $y' = x_2 \bowtie y$.*

Proof. Suppose for a contradiction that y is not a bowtie. By Lemma 5.30, y' is not a bowtie either. By definition and Axiom (F1), there exists a symp ξ through y, y' and x_1. We claim that ξ does not contain x. Indeed, suppose for a contradiction that ξ contains x. Since x is a bowtie, x' would be collinear to a line L of ξ through x. Hence y' is collinear to a point p on L, and x_1 is collinear to a point p' on $y'p$. Since $x' = x_2 \bowtie x$, we see that x is not collinear to y'. Hence $p' \neq x$. But then $x' \perp p' \perp x_1 \perp x \perp x'$, implying that x_1 and x' are symplectic, which contradicts $x = x_1 \bowtie x'$. Hence $x \notin \xi$.

Let w in ξ be collinear to x_1 and y' but not to y. Since x_1 is a bowtie, the fact that $x \notin \xi$ together with Corollary 5.26 imply that $x_1 = w \bowtie x$. Since $x = x_1 \bowtie x'$, the point w is not collinear to x'. Now assume first that y' is not a bowtie either. By definition, there exists a symp ξ' through wy' and $y'x'$. The intersection $\xi' \cap \zeta$ contains $x'y'$, and, since x' is a bowtie, it contains a singular plane π through $x'y'$ (the intersection of two symps in a dual rank 3 polar space is either empty or a line). Both w and x are collinear with a line of π, so we find a point p of π collinear to both w and x. By construction however, we had $x_1 = w \bowtie x$, but it is clear that x_1 is not contained in ζ, and hence also not in π, a contradiction. Hence y is a bowtie.

Now assume y' is a bowtie. Then there is a line K through y' collinear to x' and note that by the above $x \notin K$. In ξ, x_1 is collinear to a point z of K and so we have $x' \perp z \perp x_1 \perp x \perp x'$, again a contradiction.

The equalities $y = x_1 \bowtie y'$ and $y' = x_2 \bowtie y$ are also clear from the arguments above. The assertion is proved. ☐

Lemma 5.35. *Let x_1 and x_2 be points in X with $\delta(x_1, x_2) = 3$. If (x_1, x, x', x_2) is a 3-path with x a bowtie, then for every plane π through xx_1, there is a unique line $M \in \pi$ that is contained in a symp ξ with x'; this line M contains x. Every point of M is a bowtie, and is special to x_2. The set $\{y \bowtie x_2 \mid y \in M\}$ is a line through x', which is contained in ξ.*

Proof. The existence and uniqueness of M follows from the fact that $x = x_1 \bowtie x'$ and the property of dual rank 3 polar spaces that the set of points symplectic to a given point is a hyperplane. By Lemma 5.30, the point x' is also a bowtie, implying that x_2 is collinear with a line L of ξ. Every point y of M is collinear with some point y' of L, and the correspondence $M \to L : y \mapsto y'$ is a bijection as otherwise either x or x' is collinear to L or M, respectively, contradicting one of $x = x_1 \bowtie x'$ or $x' = x_2 \bowtie x$. By Lemma 5.34, the points y and y' are bowties, $y = x_1 \bowtie y'$, and $y' = x_1 \bowtie y$. This proves the assertions. ☐

Lemma 5.36. *Let x_1 and x_2 be points in X with $\delta(x_1, x_2) = 3$. Suppose that (x_1, x, x', x_2) is a 3-path with x a bowtie. Every line L through x_1 contains a unique point at distance 2 from x_2, which is moreover a bowtie.*

Proof. Since $\mathsf{Res}(x_1)$ is connected, it suffices to prove this for lines L through x_1 different from x_1x which are contained in a plane π_1 together with x. Lemma 5.35 yields a bowtie $y \in L$ at distance 2 from x_2. It remains to show uniqueness of y as point of L at distance 2 from x_2.

Therefore, suppose for a contradiction that there is some point z on L different from y for which $\delta(z, x_2) = 2$. In particular, there exists some point w collinear to both z and x_2.

As in Lemma 5.35, let ξ be a symp through xx' that intersects π_1 in the line xy. Denote with π_2 the plane that contains x_2 and a line of ξ. The point $y' := y \bowtie x_2$ is contained in $\xi \cap \pi_2$ and is a bowtie by Lemma 5.35. Note that w is not contained in $x'y'$ as this would imply by Lemma 5.30 that $z = w \bowtie x_1$, contradicting the fact that w is also collinear to some point of xy, see Lemma 5.35.

We first prove that w is collinear to a line of π_2. Suppose that this is not the case. By Lemma 5.33, the point x_2 is a bowtie, implying that there is some symp ξ_2 through wx_2 that intersects π_2 in a line. Let w_2' be the unique point of $x'y'$ contained in ξ_2. If w is a bowtie, the point z is collinear to some line L_z through w in ξ_2. If w is not a bowtie, then there is a symp ξ' containing w, z and x_2. Since x_2 is a bowtie, the symps ξ' and ξ_2, intersect in at least a plane (since their intersection contains the line x_2w), so we also find a line L_z through w collinear with z (inside that plane). The point w_2' is collinear with some point p_z of L_z. We claim that $p_z \notin xy$. Indeed, otherwise Lemma 5.35 implies that p_z is bowtie, and by Lemma 5.30, $w = x_2 \bowtie p_z$, contradicting $x_2 \perp p_z' \perp p_z$, with $p_z' \in x'y'$ and the fact that $w \neq p_z'$ since $w \notin x'y'$. The claim follows. Hence the unique point w_2 of xy collinear to w_2' differs from p_z and so we find a symp ξ^* through w_2, w_2' and z. However, the point w_2' is then contained in symps together with xy and also together with w_2z. Considering $\mathsf{Res}(w_2)$, Lemma 5.16 implies that there is a symp containing w_2, w_2' and x_1, contradicting the fact that w_2 is bowtie and $w_2 = w_2' \bowtie x_1$.

We hence find that w is collinear to a line of π_2, which intersects $x'y'$ in some point q_2'. Replacing w_2' with q_2' and w_2 with w in the arguments of the previous paragraph yields a symp containing q_2' and x_1, again a contradiction as above. This proves the nonexisteince of $z \neq y$ and the proof of the lemma is complete. ☐

Lemma 5.37. *Let (x_1, x_2, x_3, x_4) be a bowtie path. Then it is a 3-path, so $\delta(x_1, x_4) = 3$.*

Proof. Suppose for a contradiction that $\delta(x_1, x_4) \leq 2$. Then $\delta(x_1, x_4) = 2$ as $x_1 \perp x_4$ implies $\{x_4, x_2\} \subseteq T_{x_1} \cap T_{x_3}$, contradicting our assumptions. Now suppose that there is a symp ξ containing x_1 and x_4. Note that $x_2 \notin \xi$. As x_1 is bowtie, there is a line L_1 through x_1 in ξ coplanar with x_2. On L_1 we find a point x_3' collinear to x_4. Then $x_3' \neq x_3$ and $\{x_3, x_3'\} \subseteq T_{x_2} \cap T_{x_4}$, a contradiction.

Hence we may assume that $T_{x_1} \cap T_{x_4} = \{x\}$, $x \in X$, and note that x is a bowtie. Let ξ_{23} be any symp through x_2x_3. Looking in $\mathsf{Res}(x_2)$, we see that there exists a line L_2 in ξ_{23} through x_2 collinear with x_1. Similarly, there exists a line L_3 in ξ_{23} through x_3 collinear with x_4. As in the proof of Lemma 5.35, the correspondence $L_2 \to L_3 : y_2 \mapsto y_3 \perp y_2$ is bijective. By Lemma 5.34, all points on L_2 and L_3 are bowties. Since x_1 is a bowtie, there is a symp ξ containing x_1, x and some point $y_2 \in L_2$. Since x is a bowtie, $x_4 \notin \xi$ and there is a line L in ξ through x collinear to x_4. Since x_2 is a bowtie, $x_3 \notin \xi$ and hence the unique point x_3' of ξ on L collinear to x_2 is distinct from x_3. Thus $x_2 \perp x_3 \perp x_4 \perp x_3' \perp x_2$, a contradiction to $x_3 = x_2 \bowtie x_4$. We conclude that $\delta(x_1, x_4) = 3$. ☐

Corollary 5.38. *Let x_1 and x_2 be points in X with $\delta(x_1, x_2) = 3$. Suppose that (x_1, x, x', x_2) is a 3-path with x a bowtie, then every point of X is a bowtie.*

Proof. Let $y_1 \in X$ be arbitrary. We show by induction on $\delta(y_1, x_1)$ that y_1 is contained in a 3-path (y_1, y, y', y_2) with y a bowtie. This then implies the assertion by Lemma 5.30.

If $\delta(y_1, x_1) = 0$, then this follows from our assumption. Now assume $\delta(y_1, x_1) \geq 1$. By the induction hypothesis, we may in fact assume $y_1 \perp x_1$. Set $L = x_1 y_1$. Then, according to Lemma 5.36, there is a unique point y on L at distance 2 from x_2, and it is a bowtie. If $y_1 \neq y$, then $\delta(y, x_2) = 3$ and there is a 3-path containing y_1, y, x_2. The assertion follows from Lemma 5.33. Hence we may assume that $y_1 = y$.

Since x_1 is a bowtie, Corollary 5.27 yields a line K through x_1 such that for each point $z \in K \setminus \{x_1\}$ we have $x_1 = y \bowtie z$. By Lemma 5.36, we may assume that $z = x$. Now (y, x_1, x, x') is a bowtie path, and Lemma 5.37 implies that it is a 3-path, with x_1 a bowtie. The corollary now follows from the connectivity of X. □

5.4.2 When a point is at distance 3 from another point

Lemma 5.39. *Let x_1 and x_2 be points in X with $\delta(x_1, x_2) = 3$. Then both x_1 and x_2 are bowties.*

Proof. Suppose for a contradiction that x_1 is not a bowtie. We first show that x_2 is a bowtie. Since $\delta(x_1, x_2) = 3$, there exists at least one 3-path (x_1, x, x', x_2). By Corollary 5.38, the point x is not a bowtie. In particular, there is a symp ξ containing x_1, x and x'. Every point y in ξ collinear to both x_1 and x' is contained in a 3-path connecting x_1 and x', which, by Lemma 5.32, implies that $\dim T_y^{x_1} \leq 3$.

Now, since x' is not a bowtie (again by Corollary 5.38), there is some symp ξ' containing x, x' and x_2. Suppose, for a contradiction, that $T_x(\xi') \subseteq \mathbb{P}(\xi)$ (which is only possible if the symps are symplectic). Since $x_1 \notin \xi'$, the space $T_{x_1}(\xi) \cap T_x(\xi') \cap T_{x_2}(\xi')$ is 2-dimensional and, since the symps are symplectic (implying $T_x(\xi') \subseteq X$), we find a point $z \in X$ collinear to both x_1 and x_2, a contradiction to $\delta(x_1, x_2) = 3$.

Hence there exists a point u collinear to both x and x_2, and not contained in ξ. As x is not a bowtie, we find a symp ζ containing x_1, x and u, and clearly $\mathbb{P}(\xi) \neq \mathbb{P}(\zeta)$. Again, for each point $y \in \zeta$ collinear to both u and x_1 we have $\dim T_y^{x_1} \leq 3$. If $T_{x_1}(\xi) \neq T_{x_1}(\zeta)$, then Lemma 5.28 implies that x_1 is a bowtie.

So suppose $T_{x_1}(\xi) = T_{x_1}(\zeta)$. Let $y_1 \in xx_1 \setminus \{x, x_1\}$.

If $\delta(x_2, y_1) = 2$, then x_2 and y_1 are not special (as otherwise every point is a bowtie by Corollary 5.38) and so there is a symp ζ_1 through x_2 and y_1. If $T_{x_2}(\zeta_1) = T_{x_2}(\xi')$, then, by dimension, some point collinear to x_2 is collinear to both x and y_1, and so also to x_1, contradicting $\delta(x_1, x_2) = 3$. The argument of the first paragraph applied to ξ' and ζ_1 implies that x_2 is a bowtie. We refer to the arguments in this paragraph by (*).

So we may assume that $\delta(x_2, y_1) = 3$. Since $\mathbb{P}(\xi) \neq \mathbb{P}(\zeta)$, we deduce that $T_{y_1}(\xi) \neq T_{y_1}(\zeta)$. Hence, interchanging the roles of x_1 and y_1, we find that y_1 is a bowtie. Now let y_1 vary over $A := T_{x_1}(\xi) \setminus T_{x'}(\xi)$. Let α_i, $i = 1, 2$, be a singular plane of ξ through x_1, with $\alpha_1 \neq \alpha_2$ and pick a point $y_1 \in \alpha_1 \cap \alpha_2 \cap A$ and a point $u_1 \in (A \cap \alpha_1) \setminus \alpha_2$. Let $L \neq x_1 y_1$ be any line in α_2 through y_1. Note that L contains a unique point $a \in A$. Since y_1 is a bowtie, there is a plane β through L with the property that each point of $\beta \setminus L$ is special to u_1. Let M be a line in β containing a and not y_1. then by the previous arguments, all points of $M \setminus \{a\}$ are at distance 3 from x_2, we can let two of them play the roles of x_1 and y_1 and obtain a bowtie $w_1 \in M$. Hence we have three bowties $u_1 \perp y_1 \perp w_1$ with $y_1 = u_1 \bowtie w_1$. Completely similar there exists a bowtie $v_1 \perp u_1$ with $u_1 = y_1 \bowtie v_1$. Consequently we have a bowtie path (v_1, u_1, y_1, w_1). Lemma 5.37 implies that this is a 3-path and Corollary 5.38 then implies that all points are bowties, a contradiction.

Hence we have shown that x_2 is a bowtie. Interchanging the roles of x_1 and x_2 in (*), we deduce that the set of points at distance at most 2 from x_1 is a subspace. Moreover, since x_1 is not a bowtie, every point at distance 3 from x_1 is a bowtie, by our previous arguments. But then the arguments in the previous paragraph can be copied to prove the existence of a bowtie path starting with a plane of ξ' through x_2. This shows that x_1 is a bowtie after all.

The proof of the lemma is complete. $\square$

Lemma 5.40. *If X contains a point that is not a bowtie, then for all points $x \in X$, the points at distance at most two from x form a subspace.*

Proof. Let x be a point of X, and let L be a line containing two points y_1 and y_2 with $\delta(x, y_1) = \delta(x, y_2) = 2$. Suppose for a contradiction that there is a point y on L with $\delta(x, y) = 3$. By Lemma 5.39, both x and y are bowties. For $i = 1, 2$, let y_i' be arbitrary but such that $x \perp y_i' \perp y_i$. By Corollary 5.38, the point y_i' is not a bowtie, hence there exists a symp ξ_i containing x, y_i' and y_i, $i = 1, 2$. We claim that y_1' and y_2' are not special. To that end, denote with ζ_i a symp through L and $y_i y_i'$ (the existence of ζ_i is ensured by Corollary 5.38, which implies that y_i is not a bowtie), $i = 1, 2$. The symps ζ_1 and ζ_2 intersect in the line L. The point $y \in L$ however is a bowtie, so they intersect in a plane, which contains a point $p \neq x$ collinear to both y_1' and y_2'. This proves that $T_{y_1'} \cap T_{y_2'}$ contains both x and p, and hence the claim follows.

The point x is a bowtie, so $\mathsf{Res}(x)$ is a dual rank 3 polar space. In this residue, ξ_1 and ξ_2 correspond to two symps ξ_1' and ξ_2', respectively, such that, by the above claim and the randomness of y_i' in ξ_i collinear to x, $i = 1, 2$, every pair of points (u_1', u_2') of $\xi_1' \times \xi_2'$ is equal, collinear or symplectic. This can only be the case when ξ_1' and ξ_2' intersect in line of $\mathsf{Res}(x)$, or, in other words, when ξ_1 and ξ_2 share a singular plane α through x. But then there is a point q of α collinear to both y_1 and y_2, and hence, by Lemma 5.2, to $y \in L$. This is a contradiction to the fact that $\delta(x, y) = 3$. $\square$

Lemma 5.41. *All points of X are bowties.*

Proof. Suppose for a contradiction that not all points are bowties. Axiom (F1$'$) yields two points x and y at distance 3. By Lemma 5.39, both x and y are bowties. Let (x, q, p, y) be a 3-path. By Corollary 5.38 neither p nor q is a bowtie. This yields symps ξ and ζ containing x, q, p and y, p, q, respectively. Let y' be a point of ζ collinear to y and not collinear to either p or q. Then $y' \perp p'$, with $p' \in pq \setminus \{p, q\}$. Next, let q' be a point of ξ collinear to both p' and x, but not to p.

The points at distance 2 from y form a subspace S by Lemma 5.40. Then S intersects ξ in a subspace containing all points collinear to p in ξ, and not containing x. Hence $S \cap \mathbb{P}(\xi) = T_p(\xi)$. It follows that $\delta(y, q') = 3$ and hence q' is a bowtie. Likewise, $\delta(x, y') = 3$. But then (y', p', q', x) is a 3-path with q' a bowtie, implying by Corollary 5.38 that all points are bowties after all. $\square$

5.5 Identifying the geometry

Define the following incidence geometry $\mathscr{G}(X)$ with objects of type 1 up to 4. The objects of type 1 are the symps of (X, Ξ), the ones of type 2 are the singular planes, the type 3 objects are the singular lines, and, finally, the objects of type 4 are the points of X.

Incidence is containment made symmetric. We show that the diagram of this geometry is F_4, where we have chosen the types above so that they conform to the Bourbaki labeling [4].

By Corollary 5.26 and Lemma 5.41, the residue at each point is isomorphic to $\mathsf{B}_{3,3}(\mathbb{K},\mathbb{A})$, for some field $\mathbb{K}$ over which $\mathbb{A}$ is a quadratic associative division algebra. Whence we know all rank 2 residues of type $\{i,j\}$, with $\{i,j\} \subseteq \{1,2,3\}$. Also, in the same way, the residues of type $\{2,3,4\}$ of $\mathscr{G}(X)$ correspond to the geometry of the symps, which are polar spaces isomorphic to $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$. This establishes all rank 2 residues of type $\{i,j\}$, for all $\{i,j\} \subseteq \{2,3,4\}$. It remains to check the residues of type $\{1,4\}$. But these are trivially all generalized digons.

Now we want to apply Proposition 9 of Tits [31]. Hence we have to verify the following four properties of X:

(LL) *If two singular lines are both incident to two distinct points, they coincide.* This is trivially true since we are working in a projective space.

(LH) *If a line and a symp are both incident to two distinct points, they are incident.* This is also trivially true by working in a projective space.

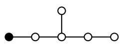
(HH) *If two distinct symps are both incident to two distinct points, the latter are incident to a line.* This follows from the convexity that we proved.

(O) *If two lines contain the same point set, they coincide.* This follows from (LL)

It now follows that $\mathscr{G}(X)$ is a geometry isomorphic to $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$, associated to the building $\mathsf{F}_4(\mathbb{K},\mathbb{A})$ and the proof of Theorem B is complete.

6 About admissible quotients

By definition, an admissible quotient of the universal embedding of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ is an injective projection from a subspace such that the Axioms (F1), (F1′), (F2) and (F3) hold. We show with an example that these exist. We need some preparation, but our treatment will be rather sketchy (motivated by the fact that this is not the essential part of our results).

We begin with defining a certain variety denoted $\mathscr{F}_{4,4}(\mathbb{K})$, which is the universal embedding of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$ in $\mathsf{PG}(25,\mathbb{K})$. This is done by intersecting another variety, denoted $\mathscr{E}_{6,1}(\mathbb{K})$, in $\mathsf{PG}(26,\mathbb{K})$ with a hyperplane. The latter variety is the universal embedding of the minuscule geometry $\mathsf{E}_{6,1}(\mathbb{K})$  related to the building $\mathsf{E}_6(\mathbb{K})$. It is defined as follows (see [35], which is based on [2] and to which we refer for undefined notions here; some ideas also stem from [25]).

Let $\Gamma = (X,\mathscr{L})$ be the unique generalized quadrangle of order $(2,4)$. Let V be the 27-dimensional vector space over $\mathbb{K}$ whose standard basis is labeled by the points of Γ, that is, the standard basis of V can be written as $\{e_p \mid p \in X\}$. Let $\mathscr{S}$ be a Hermitian spread of Γ. Each point $p \in X$ defines a unique quadratic form $Q_p : V \to \mathbb{K}$ given in coordinates by

$$Q_p(v) \quad = \quad x_{q_1}x_{q_2} - \sum_{\{p,r_1,r_2\}\in\mathscr{L}\setminus\mathscr{S}} x_{r_1}x_{r_2}, \qquad (1)$$

where $v = (x_r)_{r\in X}$ and $\{p,q_1,q_2\} \in \mathscr{S}$. Now define the map $\phi : V \to V : v \mapsto (Q_p(v))_{p\in X}$. Then $\phi(\phi(v)) = \mathfrak{C}(v)v$, where

$$\mathfrak{C}(v) = \sum_{\{p,q,r\}\in\mathscr{S}} x_px_qx_r - \sum_{\{p,q,r\}\in\mathscr{L}\setminus\mathscr{S}} x_px_qx_r$$

is a cubic form and $\phi(v) = \nabla\mathfrak{C}(v)$ (the gradient in the classical sense). Denoting the ordinary dot or inner product of two vectors v and w by $v.w$, we have the identity

$$\mathfrak{C}(v+\lambda w) = \mathfrak{C}(v) + \lambda\phi(v).w + \lambda^2 v.\phi(w) + \lambda^3\mathfrak{C}(w),$$

for all $v, w \in V$ and $\lambda \in \mathbb{K}$. Note also that $v.w = v^*(w) = w^*(v)$.

Following the terminology in [2], let us call a point $\langle v \rangle$, $v \in V^\times$, of $\mathsf{PG}(V)$

- (i) *white* if $\phi(v) = \vec{o}$;
- (ii) *gray* if $\mathfrak{C}(v) = 0$ and $\phi(v) \neq \vec{o}$;
- (iii) *black* if $\mathfrak{C}(v) \neq 0$.

The set of white points is the co-called *exceptional variety* $\mathscr{E}_{6,1}(\mathbb{K})$.

Let V^* be the dual vector space and let $\{f_p \mid p \in X\}$ be the corresponding dual basis. Every vector $v = \sum x_p e_p$ corresponds to its *dual vector* $v^* = \sum x_p f_p$, and each dual vector v^* on its turn defines a unique hyperplane $H(v)$ of the projective space $\mathsf{PG}(V)$ corresponding to V (and consisting of the points $\langle w \rangle$ with $v^*(w) = v.w = 0$). Let $\langle b \rangle$ be a black point, then $H(b)$ intersects $\mathscr{E}_{6,1}(\mathbb{K})$ in (a copy of) $\mathscr{F}_{4,4}(\mathbb{K})$. Let us call a hyperplane $H(v)$ *white, grey, black* according to whether the corresponding point $\langle v \rangle$ is white, grey or black, respectively. White hyperplanes H have the characterizing property of intersecting $\mathscr{E}_{6,1}(\mathbb{K})$ in a hyperplane of $\mathsf{E}_{6,1}(\mathbb{K})$ consisting of a unique symp $\xi(H)$ and all points of $\mathsf{E}_{6,1}(\mathbb{K})$ collinear to some point of $\xi(H)$.

Concretely, let $\{p_1, p_2, p_3\}$ be a member of $\mathscr{S}$. Then $b = e_{p_1} + e_{p_2} + e_{p_3}$ is a black point with $\phi(b) = b$ and $\mathfrak{C}(b) = 1$. We now assume $\mathsf{char}\,\mathbb{K} = 3$. Then the hyperplane $H(b)$ contains the point $\langle b \rangle$. We claim that the projection of $\mathscr{F}_{4,4}(\mathbb{K})$, obtained as the intersection $\mathscr{E}_{6,1}(\mathbb{K}) \cap H(b)$, from $\langle b \rangle$ is admissible. In order to sketch a proof of this, we note that the following polarity ρ of $\mathsf{PG}(26, \mathbb{K})$, given by its action on the basis of V, induces a polarity, also denoted ρ, of $\mathsf{E}_{6,1}(\mathbb{K})$ (interchanging points and symps) the set of absolute points (that is, points lying in their image) of which is exactly $\mathscr{F}_{4,4}(\mathbb{K})$.

$$\rho : V \to V^* : e_p \mapsto \begin{cases} f_p \text{ if } p \in \{p_1, p_2, p_3\} \\ f_q \text{ if } p \notin \{p_1, p_2, p_3\} \text{ and } \{p, q, p_i\} \in \mathscr{L} \text{ for some } i \in \{1, 2, 3\} \end{cases}$$

For an absolute point $x \in \mathscr{F}_{4,4}(\mathbb{K})$, the symp $\xi(x^\rho)$ intersects $\mathscr{E}_{6,1}(\mathbb{K})$ precisely in $x^\perp \cap \xi(x^\rho)$ (where $\perp$ stands for collinearity in $\mathsf{E}_{6,1}(\mathbb{K})$) and this intersection is precisely the set of points of $\mathscr{F}_{4,4}(\mathbb{K})$ collinear to x (in $\mathsf{F}_{4,4}(\mathbb{K})$). We also call these symps $\xi(x^\rho)$ *absolute*.

Each symp ξ of $\mathsf{E}_{6,1}(\mathbb{K})$ can be seen through gray points in $\mathscr{E}_{6,1}(\mathbb{K})$ as follows. There exists a white point $\langle w \rangle$ such that the gray points of $\langle \xi \rangle$ are precisely those points $\langle v \rangle$ for which $\phi(v) \in \mathbb{K}w$. We denote $\langle w \rangle = c(\xi)$. Then, moreover, for each symp ξ of $\mathsf{E}_{6,1}(\mathbb{K})$, we have that ξ, as a quadric in $\mathscr{E}_{6,1}(\mathbb{K})$, is absolute if and only if $c(\xi)$ is absolute (but ξ might be different from $\xi(c(\xi)^\rho)$).

Now we note that the projection X of $\mathscr{F}_{4,4}(\mathbb{K})$ from any black point satisfies (F1), (F2) and (F3). It is (F1$'$) we are concerned with, and in particular the part where the members of Π intersect X in a pair of lines and nothing more, the tangent subspaces of two points are disjoint if these points are opposite in $\mathsf{F}_{4,4}(\mathbb{K})$, and they intersect in a unique point if these points are special in $\mathsf{F}_{4,4}(\mathbb{K})$. All of these obstructions are killed if we show the nonexistence of a line L in $H(b)$ containing $\langle b \rangle$ and two distinct points $\langle w_1 \rangle$ and $\langle w_2 \rangle$ such that $\phi(w_i) = \lambda_i v_i$, with $\lambda_i \in \mathbb{K}$ and $\langle v_i \rangle$ an absolute point (hence contained in $H(b)$), $i = 1, 2$. By redefining λ_i if necessary, we may assume $b = w_1 + w_2$. Since $\langle v_i \rangle$ belongs to $\mathscr{F}_{4,4}(\mathbb{K})$, it is a white point and so $\phi(v_i) = 0$ ($\langle w_i \rangle$ could be gray or white), $i = 1, 2$. This implies $\mathfrak{C}(w_i) = 0$, $i = 1, 2$. Using $\phi(b) = b$ and $\mathfrak{C}(b) = 1$, we derive from the above identity

$$0 = \mathfrak{C}(w_2) = \mathfrak{C}(b - w_1) = \mathfrak{C}(b) - \phi(b).w_1 + \phi(w_1).b - \mathfrak{C}(w_1) = 1 - b.w_1 - \lambda_1 b.v_1 = 1,$$

clearly a contradiction. Hence we established our example.

Remark 6.1. It is tempting to conjecture that the above is essentially the only example of a non-trivial admissible projection; however this is not entirely clear to us, and especially the cases $\mathbb{A} \neq \mathbb{K}$ seem rather hopeless at the moment. A safer conjecture would be that an admissible projection is always from a subspace of dimension at most 1 (a point or the empty subspace), stemming from our inability to find an admissible projection from a line.

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