The Hermitian variety $H(5, 4)$ has no ovoid

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The Hermitian variety $H(d, q^2)$ is the set of points of $PG(d, q^2)$ satisfying the equation

$$X_0^{q+1} + X_1^{q+1} + \ldots X_d^{q+1} = 0$$

When $d = 2n + 1, 2n$ respectively, $H(d, q^2)$ contains points, lines, \ldots, $n$-dimensional subspaces of $PG(d, q^2)$, $(n - 1)$-dimensional subspaces of $PG(d, q^2)$ respectively.

The Hermitian variety $H(d, q^2)$ is a example of a so-called classical polar space. The subspaces of maximal dimension are also called generators.
An ovoid of a Hermitian variety $H(d, q^2)$ is a set $O$ of points of $H(d, q^2)$ such that every generator meets $O$ in exactly one point.
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If $H(d - 2, q^2)$ has no ovoids, then $H(d, q^2)$ has no ovoids.
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Known results

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G.E. Moorhouse: $H(2n + 1, q^2)$, $q = p^h$, $p$ prime, $h > 1$ has no ovoids if

$$p^{2n+1} > \left(\frac{2n + p}{2n + 1}\right)^2 - \left(\frac{2n + p - 1}{2n + 1}\right)^2$$

Ovoids of $H(5, q^2)$ are not excluded.
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A. Klein: $H(2n + 1, q^2)$ has no ovoids if $n > q^3$. 
Suppose that $\mathcal{O}$ is an ovoid of $\mathbb{H}(3, 4)$. There exists a plane $\pi$, $\pi \cap \mathbb{H}(3, 4) = \mathbb{H}(2, 4)$, such that either

1. $\pi \cap \mathbb{H}(3, 4) = \mathbb{H}(2, 4) = \mathcal{O}$, or

2. $\mathcal{O} = (\mathbb{H}(2, 4) \setminus L) \cup (L^\perp \cap \mathbb{H}(3, 4))$, $L$ a line of $\pi$, $L \cap \mathbb{H}(3, 4) = \mathbb{H}(1, 4)$. 
Suppose that \( \mathcal{O} \) is an ovoid of \( \text{H}(3, 4) \). There exists a plane \( \pi \), \( \pi \cap \text{H}(3, 4) = \text{H}(2, 4) \), such that either

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Every partial ovoid of \( \text{H}(3, 4) \) containing 8 points can be extended to an ovoid of \( \text{H}(3, 4) \).
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Every plane $\pi$ meets $\mathcal{O}$ in 0, 1, 2, 3 or 6 points.
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If $\pi$ is a plane, $|\pi \cap \mathcal{O}| = 3$, then the points of $\pi \cap \mathcal{O}$ are collinear.
Suppose that $\mathcal{O}$ is an ovoid of $H(5, 4)$. Let $p$ be a point of $H(5, 4) \setminus \mathcal{O}$. Then $|p^\perp \cap \mathcal{O}| = 9$. If $\pi$ is a plane in $p^\perp$, $\pi \cap H(5, 4) = H(2, 4)$, then $|\langle p, \pi \rangle| \in \{0, 1, 2, 3, 6, 9\}$
Suppose that $\mathcal{O}$ is an ovoid of $H(5, 4)$. Let $p$ be a point of $H(5, 4) \setminus \mathcal{O}$. Then $|p^\perp \cap \mathcal{O}| = 9$. If $\pi$ is a plane in $p^\perp$, $\pi \cap H(5, 4) = H(2, 4)$, then $|\langle p, \pi \rangle| \in \{0, 1, 2, 3, 6, 9\}$

Suppose that $\mathcal{O}$ is an ovoid of $H(5, q^2)$. Consider a plane $\pi$ that meets the variety in $H(2, q^2)$ and put $m := |\pi \cap \mathcal{O}|$.

Suppose furthermore that $1 \leq m < q^3 + 1$. Let $A$, resp. $B$, be the set consisting of all points $x \in \mathcal{O} \setminus \pi$ such that $\langle \pi, x \rangle$ meets $H(5, q^2)$ in a cone $sH(2, q^2)$, resp. an $H(3, q^2)$.

- We have $|A| = (q^2 - 1)(q^2 - 1 + m)$ and $|B| = q^2(q^3 - q^2 + 2 - m)$.

- If $q = 2$ and $x$ is a point of $(\pi \cap H(5, 4)) \setminus \mathcal{O}$, then $|x^\perp \cap B| \in \{0, 3, 6, 7, 8, 9\}$. 
Suppose that $\mathcal{O}$ is an ovoid of $H(5, 4)$. Then $|\pi \cap \mathcal{O}| \leq 3$ for every plane $\pi$, $\pi \cap H(5, 4) = H(2, 4)$ and $|\alpha \cap \mathcal{O}| < 6$ for every 3-dimensional space $\alpha$, $\alpha \cap H(5, 4) = H(3, 4)$. 

The last steps

The Hermitian variety $H(5, 4)$ has no ovoid – p.7/8
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Suppose that \( O \) is an ovoid of \( H(5, 4) \). Then \(|\pi \cap O| \leq 3\) for every plane \( \pi \), \( \pi \cap H(5, 4) = H(2, 4) \) and \(|\alpha \cap O| < 6\) for every 3-dimensional space \( \alpha \), \( \alpha \cap H(5, 4) = H(3, 4) \).

The Hermitian variety \( H(5, 4) \) has no ovoids
References


