Characterization results on arbitrary (weighted) minihypers and linear codes meeting the Griesmer bound

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Definition
Consider the projective plane $\text{PG}(2, q)$. A set $B$ of points of $\text{PG}(2, q)$, different from a line, is called a blocking set if any line of $\text{PG}(2, q)$ contains at least one point of $B$.

Definition
A blocking set $B$ of $\text{PG}(2, q)$ is called minimal if it does not contain a smaller blocking set as a subset.

Examples:
- The projective triangle
- A Baer subplane
- A Hermitian curve
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More blocking sets

Definition

Consider the projective space $\text{PG}(n, q)$. A set $B$ of points of $\text{PG}(2, q)$, different from a line, is called a **blocking set** if any hyperplane of $\text{PG}(n, q)$ contains at least one point of $B$. A blocking set $B$ of $\text{PG}(n, q)$ is called **minimal** if it does not contain a smaller blocking set as a subset.

Definition

Consider the projective plane $\text{PG}(2, q)$. A set $B$ of points of $\text{PG}(2, q)$ is called a **$t$-fold blocking set** if any line of $\text{PG}(2, q)$ contains at least $t$ points of $B$. A $t$-fold blocking set $B$ of $\text{PG}(2, q)$ is called **minimal** if it does not contain a smaller $t$-fold blocking set as a subset.
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Consider the projective space $\text{PG}(n, q)$. A weighted $\{f, m; n, q\}$-minihyper, $f \geq 1$, $n \geq 2$, is a pair $(F, w)$, where $F$ is a subset of the point set of $\text{PG}(n, q)$ and where $w$ is a weight function $w: \text{PG}(n, q) \rightarrow \mathbb{N}: x \mapsto w(x)$, satisfying:

1. $w(x) > 0 \iff x \in F,$
2. $\sum_{x \in F} w(x) = f,$ and
3. $\min\{\sum_{x \in H} w(x) \parallel H \in \mathcal{H}\} = m,$ where $\mathcal{H}$ is the set of hyperplanes of $\text{PG}(n, q)$. 

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Constructions . . .
**Definition**

A *linear* \([n, k, d]\)-code \(C\) over the finite field \(\mathbb{GF}(q)\) is a \(k\)-dimensional subspace of the \(n\)-dimensional vector space \(V(n, q)\), where \(d\) is the *minimum distance* of \(C\).

**Theorem**

Suppose that \(C\) is a linear \([n, k, d]\) code. The Griesmer bound states that

\[
  n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d'}{q^i} \right\rceil = g_q(k, d),
\]

where \(\lceil x \rceil\) denotes the smallest integer greater than or equal to \(x\).
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where $\left\lceil x \right\rceil$ denotes the smallest integer greater than or equal to $x$. 
Suppose that $C$ is a linear $[n, k, d]$ code. Then we can write $d$ in an unique way as $d = \theta q^{k-1} - \sum_{i=0}^{k-2} \epsilon_i q^\lambda_i$ such that $\theta \geq 1$ and $0 \leq \epsilon_i < q$. Then the Griesmer bound for an $[n, k, d]$-code can be expressed as:

$$n \geq \theta v_k - \sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i+1}$$

where $v_l = (q^l - 1)/(q - 1)$, for any integer $l \geq 0$. 
Linear codes meeting the Griesmer bound and minihypers

Theorem

(Hamada and Helleseth) There is a one-to-one correspondence between the set of all non-equivalent \([n, k, d]\)-codes meeting the Griesmer bound and the set of all projectively distinct \(\{\sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i+1}, \sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i}; k - 1, q\}\)-minihypers \((F, w)\), such that \(1 \leq w(p) \leq \theta\) for every point \(p \in F\).

The link is described explicitly
Introduction

Characterizations

Geometry

Linear codes

Linear codes meeting the Griesmer bound and minihypers

Let $G = (g_1 \cdots g_n)$ be a generator matrix for a linear $[n, k, d]$-code, meeting the Griesmer bound. We look at a column of $G$ as being the coordinates of a point in $PG(k - 1, q)$. Let the point set of $PG(k - 1, q)$ be $\{s_1, \ldots, s_{v_k}\}$. Let $m_i(G)$ denote the number of columns in $G$ defining $s_i$. Let $m(G)$ be the maximum value in $\{m_i(G) \mid i = 1, 2, \ldots, v_k\}$. Then $\theta = m(G)$ is uniquely determined by the code $C$ and we call it the maximum multiplicity of the code. Define the weight function $w : PG(k - 1, q) \to \mathbb{N}$ as $w(s_i) = \theta - m_i(G)$, $i = 1, 2, \ldots, v_k$. Let $F = \{s_i \in PG(k - 1, q) \mid w(s_i) > 0\}$, then $(F, w)$ is a $\{\sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i+1}, \sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i}; k - 1, q\}$-minihyper with weight function $w$. 
Some characterizations

Theorem

A weighted $t$-fold blocking set $B$ of $\text{PG}(2, q)$, $q \geq 4$, $2 \leq t < \sqrt{q} + 1$ containing no line, has at least $tq + \sqrt{tq} + 1$ points.

Theorem

A weighted $t$-fold blocking set $B$ of $\text{PG}(2, q)$ containing at least one point of weight one, of size $|B| = t(q + 1) + r$, $t + r \leq \delta_0$, contains a line.
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Table for $\delta_0$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$h$</th>
<th>$\delta_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>even</td>
<td>$\leq \sqrt{q}$</td>
</tr>
<tr>
<td>$p$</td>
<td>$h = 1$</td>
<td>$\leq (p + 1)/2$</td>
</tr>
<tr>
<td>$p$</td>
<td>3</td>
<td>$\leq p^2$</td>
</tr>
<tr>
<td>2</td>
<td>$6m + 1$, $m \geq 1$</td>
<td>$\leq 2^{4m+1} - 2^{4m} - 2^{2m+1}/2$</td>
</tr>
<tr>
<td>$&gt; 2$</td>
<td>$6m + 1$, $m \geq 1$</td>
<td>$\leq p^{4m+1} - p^{4m} - p^{2m+1}/2 + 1/2$</td>
</tr>
<tr>
<td>2</td>
<td>$6m + 3$, $m \geq 1$</td>
<td>$&lt; 2^{4m+5/2} - 2^{4m+1} - 2^{2m+1} + 1$</td>
</tr>
<tr>
<td>$&gt; 2$</td>
<td>$6m + 3$, $m \geq 1$</td>
<td>$\leq p^{4m+2} - p^{2m+2} + 2$</td>
</tr>
<tr>
<td>$\geq 5$</td>
<td>$6m + 5$, $m \geq 0$</td>
<td>$&lt; p^{4m+7/2} - p^{4m+3} - p^{2m+2}/2 + 1$</td>
</tr>
</tbody>
</table>
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Theorem

A weighted \( \{\epsilon_1(q + 1) + \epsilon_0, \epsilon_1; k - 1, q\}\)-minihyper \((F, w)\), \(k \geq 4\), with \(\epsilon_1 + \epsilon_0 \leq \delta_0\), is a sum of \(\epsilon_1\) lines and \(\epsilon_0\) points.
Theorem

(Hamada and Helleseth) A \(\mu\)-dimensional subspace intersects a weighted \(\{\sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=0}^{k-2} \epsilon_i v_i; k - 1, q\}\)-minihyper, \(\sum_{i=0}^{k-2} \epsilon_i = \delta \leq q\), \((\epsilon_0, \ldots, \epsilon_{k-2}) \in E_{\text{ext}}(k - 1, q)\), in a weighted \(\{\sum_{i=0}^{\mu} m_i v_{i+1}, \sum_{i=0}^{\mu} m_i v_i; \mu, q\}\)-minihyper, where \(\sum_{i=0}^{\mu} m_i \leq \delta\).

Theorem

Let \(F\) be a \(\{\sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=1}^{k-2} \epsilon_i v_i; k - 1, q\}\)-minihyper where \(t \geq 2\), \(q > h\), \(0 \leq \epsilon_i \leq q - 1\), \(\sum_{i=0}^{k-2} \epsilon_i = h\). Then a plane of \(\text{PG}(k - 1, q)\) is either contained in \(F\) or intersects \(F\) in an \(\{m_0 + m_1(q + 1), m_1; 2, q\}\)-minihyper with \(m_0 + m_1 \leq h\).
(Hamada and Helleseth) A $\mu$-dimensional subspace intersects a weighted $\{\sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=0}^{k-2} \epsilon_i v_i; k-1, q\}$-minihyper, 
$\sum_{i=0}^{k-2} \epsilon_i = \delta \leq q$, $(\epsilon_0, \ldots, \epsilon_{k-2}) \in E_{\text{ext}}(k-1, q)$, in a weighted $\{\sum_{i=0}^{\mu} m_i v_{i+1}, \sum_{i=0}^{\mu} m_i v_i; \mu, q\}$-minihyper, where $\sum_{i=0}^{\mu} m_i \leq \delta$.

Let $F$ be a $\{\sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=1}^{k-2} \epsilon_i v_i; k-1, q\}$-minihyper where $t \geq 2$, $q > h$, $0 \leq \epsilon_i \leq q-1$, $\sum_{i=0}^{k-2} \epsilon_i = h$.

Then a plane of $\text{PG}(k-1, q)$ is either contained in $F$ or intersects $F$ in an $\{m_0 + m_1(q+1), m_1; 2, q\}$-minihyper with $m_0 + m_1 \leq h$. 
Characterizations using planes

Theorem

A weighted \{\epsilon_2(q^2 + q + 1) + \epsilon_1(q + 1) + \epsilon_0, \epsilon_2(q + 1) + \epsilon_1; k - 1, q\}-minihyper 
\((F, w)\), with \(\epsilon_2 + \epsilon_1 + \epsilon_0 \leq \delta_0\), is a sum of \(\epsilon_2\) planes, \(\epsilon_1\) lines, and \(\epsilon_0\) points.

Theorem

A weighted \(\{\sum_{i=0}^{t} \epsilon_i v_{i+1}, \sum_{i=1}^{t} \epsilon_i v_i; k - 1, q\}\)-minihyper, with 
\(\sum_{i=0}^{t} \epsilon_i \leq \delta_0\), is the sum of \(\epsilon_t\) \(t\)-dimensional subspaces, \(\epsilon_{t-1}\) \((t - 1)\)-dimensional subspaces, \ldots, \(\epsilon_1\) lines and \(\epsilon_0\) points.
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Theorem

A weighted \( \{\sum_{i=0}^{t} \epsilon_i v_{i+1}, \sum_{i=1}^{t} \epsilon_i v_i; k - 1, q\} \)-minihyper, with \( \sum_{i=0}^{t} \epsilon_i \leq \delta_0 \), is the sum of \( \epsilon_t \) \( t \)-dimensional subspaces, \( \epsilon_{t-1} \) \( (t - 1) \)-dimensional subspaces, \ldots, \( \epsilon_1 \) lines and \( \epsilon_0 \) points.
Theorem

A union of $\epsilon_{k-2}$ $(k - 2)$-dimensional spaces, $\epsilon_{k-3}$ $(k - 3)$-dimensional spaces, ..., $\epsilon_1$ lines, and $\epsilon_0$ points, which all are pairwise disjoint, exists in $\text{PG}(k - 1, q)$, if and only if there exists a linear $[v_k - \sum_{i=0}^{k-2} \epsilon_i v_{i+1}, k, q^{k-1} - \sum_{i=0}^{k-2} \epsilon_i q^i]$-code meeting the Griesmer bound.