MEASURES AND THE DISTRIBUTIONAL $\phi-$TRANSFORM

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Abstract. We give a characterization of measures in terms of the boundary behavior of the $\phi-$transform, and obtain results on the almost everywhere convergence of the $\phi-$transform at the boundary.

1. Introduction

The aim of this article is to use the distributional $\phi-$transform, introduced in [8] in the one variable case, and here in the multidimensional case, in order to characterize the (positive) measures that belong to the distribution space $\mathcal{D}'(\mathbb{R}^n)$.

We use the notation $\mathbb{H} = \{(\mathbf{x}, t) : \mathbf{x} \in \mathbb{R}^n \text{ and } t > 0\}$. Let $F(\mathbf{x}, t)$, $(\mathbf{x}, t) \in \mathbb{H}$, be the $\phi-$transform of a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$, namely $F(\mathbf{x}, t) = \langle f(\mathbf{x} + t\mathbf{y}), \phi(\mathbf{y})\rangle$, where ϕ is a fixed *positive* test function of the space $\mathcal{D}(\mathbb{R}^n)$. We prove that f is a measure if and only if the inferior limit of $F(\mathbf{x}, t)$, as $(\mathbf{x}, t)$ approaches *any* point in the boundary $\partial\mathbb{H} = \mathbb{R}^n \times \{0\}$, in an angular fashion, is positive. Since any measure is equal to a function almost everywhere, this result provides a technique to show the existence of the almost everywhere angular limits of the $\phi-$transform of a distribution.

The plan of the article is as follows. We start by giving some necessary background in Section 2, and then continue by proving some useful properties of the multidimensional $\phi-$transform in Section 3. Then we consider the characterization of measures in Section 4.

2. Preliminaries

We shall use the notion of the distributional point value of generalized functions introduced by Łojasiewicz, in one [5] and several variables [6]. Let $f \in \mathcal{D}'(\mathbb{R}^n)$, and let $\mathbf{x}_0 \in \mathbb{R}^n$. We say that f has the distributional point value γ at $\mathbf{x} = \mathbf{x}_0$, and write

$$f(\mathbf{x}_0) = \gamma, \quad \text{distributionally}, \tag{2.1}$$

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if $\lim_{\varepsilon\to 0} f(\mathbf{x}_0+\varepsilon\mathbf{x})=\gamma$ in the space $\mathcal{D}'(\mathbb{R}^n)$, that is, if

$$\lim_{\varepsilon\to 0}\left\langle f(\mathbf{x}_0+\varepsilon\mathbf{x}),\phi(\mathbf{x})\right\rangle=\gamma\int_{\mathbb{R}^n}\phi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,, \tag{2.2}$$

for all test functions $\phi\in\mathcal{D}(\mathbb{R}^n)$. It can be shown that $f(\mathbf{x_0})=\gamma$, distributionally, if and only if there exists a multi-index $\mathbf{k}_0\in\mathbb{N}^n$ such that for all multi-indices $\mathbf{k}\geq\mathbf{k}_0$ there exists a $\mathbf{k}$ primitive of f, G with $\mathbf{D}^{\mathbf{k}}G=f$, that is a continuous function in a neighborhood of $\mathbf{x}=\mathbf{x}_0$ and satisfies

$$G(\mathbf{x})=\frac{\gamma(\mathbf{x}-\mathbf{x}_0)^{\mathbf{k}}}{\mathbf{k}!}+o\left(|\mathbf{x}-\mathbf{x}_0|^{|\mathbf{k}|}\right)\,,\quad\text{as } \mathbf{x}\to\mathbf{x}_0\,. \tag{2.3}$$

It is important to observe that the distributional point values determine a distribution if they exist *everywhere,* that is, if $f\in\mathcal{D}'(\mathbb{R}^n)$ is such that $f(\mathbf{x_0})=0$ distributionally $\forall\mathbf{x_0}\in\Omega$, where Ω is an open set, then $f=0$ in Ω [5, 6].

We shall follow [2, 4] for the notions related to Cesàro behavior of distributions. If $f\in\mathcal{D}'(\mathbb{R}^n)$ and $\alpha\in\mathbb{R}$ is not a negative integer, we say that f is bounded by $|\mathbf{x}|^{\alpha}$ in the Cesàro sense for $|\mathbf{x}|$ large, and write

$$f(\mathbf{x})=O(|\mathbf{x}|^{\alpha})\quad(\mathrm{C})\,,\quad\text{as } |\mathbf{x}|\to\infty\,, \tag{2.4}$$

if there exists a multi-index $\mathbf{k}\in\mathbb{N}^n$ and a $\mathbf{k}$ primitive, $\mathbf{D}^{\mathbf{k}}G=f$, which is a (locally integrable) function for $|\mathbf{x}|$ large and satisfies the *ordinary* order relation

$$G(\mathbf{x})=O\left(|\mathbf{x}|^{\alpha+|\mathbf{k}|}\right)\,,\quad\text{as } |\mathbf{x}|\to\infty\,. \tag{2.5}$$

Naturally (2.5) will not hold for all primitives of f, and if it holds for $\mathbf{k}$ it will also hold for bigger multi-indices.

3. The $\phi-$transform

In this section we explain how we can extend to several variables the $\phi-$transform introduced in [8] (see also [1]). Let $\phi\in\mathcal{D}(\mathbb{R}^n)$ be a fixed *normalized* test function, that is, one that satisfies

$$\int_{\mathbb{R}^n}\phi(\mathbf{x})\,\mathrm{d}\mathbf{x}=1\,. \tag{3.1}$$

If $f\in\mathcal{D}'(\mathbb{R}^n)$ we introduce the function of $n+1$ variables $F=F_{\phi}\{f\}$ by the formula

$$F(\mathbf{x},t)=\left\langle f(\mathbf{x}+t\mathbf{y}),\phi(\mathbf{y})\right\rangle\,, \tag{3.2}$$

where $(\mathbf{x},t)\in\mathbb{H}$, the half space $\mathbb{R}^n\times(0,\infty)$. Naturally the evaluation in (3.2) is with respect to the variable $\mathbf{y}$. We call F the $\phi-$transform of f. Whenever we consider $\phi-$transforms we assume that ϕ satisfies (3.1).

The definition of the $\phi-$transform tell us that if $f(\mathbf{x}_0) = \gamma$, then $F(\mathbf{x}_0, t) \to \gamma$ as $t \to 0^+$, but actually $F(\mathbf{x}, t) \to \gamma$ as $(\mathbf{x}, t) \to (\mathbf{x}_0, 0)$ in an *angular* or *non-tangential* fashion, that is if $|\mathbf{x} - \mathbf{x}_0| \leq Mt$ for some $M > 0$ (just replace $\phi(\mathbf{x})$ in (2.2) by $\phi(\mathbf{x} - r\omega)$ where $|\omega| = 1$ and $0 \leq r \leq M$.)

We can also consider the $\phi-$transform if $\phi \in \mathcal{A}(\mathbb{R}^n)$ satisfies (3.1) and $f \in \mathcal{A}'(\mathbb{R}^n)$, where $\mathcal{A}(\mathbb{R}^n)$ is a suitable space of test functions, such as $\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{K}(\mathbb{R}^n)$.

We start with the distributional convergence of the $\phi-$transform.

Theorem 1. *If $\phi \in \mathcal{D}(\mathbb{R}^n)$ and $f \in \mathcal{D}'(\mathbb{R}^n)$, then*

$$\lim_{t \to 0^+} F(\mathbf{x}, t) = f(\mathbf{x}) , \tag{3.3}$$

distributionally in the space $\mathcal{D}'(\mathbb{R}^n)$, that is, if $\rho \in \mathcal{D}(\mathbb{R}^n)$ then

$$\lim_{t \to 0^+} \langle F(\mathbf{x}, t), \rho(\mathbf{x}) \rangle = \langle f(\mathbf{x}), \rho(\mathbf{x}) \rangle . \tag{3.4}$$

Proof. We have that

$$\langle F(\mathbf{x}, t), \rho(\mathbf{x}) \rangle = \langle \varrho(t\mathbf{y}), \phi(\mathbf{y}) \rangle , \tag{3.5}$$

where

$$\varrho(\mathbf{z}) = \langle f(\mathbf{x}), \rho(\mathbf{x} - \mathbf{z}) \rangle , \tag{3.6}$$

is a smooth function of $\mathbf{z}$. The Łojasewicz point value $\varrho(\mathbf{0})$ exists and equals the ordinary value and thus

$$\lim_{t \to 0^+} \langle \varrho(t\mathbf{y}), \phi(\mathbf{y}) \rangle = \varrho(\mathbf{0}) = \langle f(\mathbf{x}), \rho(\mathbf{x}) \rangle , \tag{3.7}$$

as required. □

The result of the Theorem 1 also hold in other cases. In order to obtain those results we need some lemmas. Recall that an asymptotic order relation is *strong* if it remains valid after differentiation of any order.

Lemma 1. *Let $f \in \mathcal{E}'(\mathbb{R}^n)$ be a distribution with compact support K. Let $\phi \in \mathcal{E}(\mathbb{R}^n)$ be a test function that satisfies (3.1) and*

$$\phi(\mathbf{x}) = O\left(|\mathbf{x}|^\beta\right), \quad \text{strongly as } |\mathbf{x}| \to \infty , \tag{3.8}$$

where $\beta < -n$. Then

$$\lim_{t \to 0^+} F(\mathbf{x}, t) = 0 , \tag{3.9}$$

uniformly on compacts of $\mathbb{R}^n \setminus K$.

Proof. There exits a constants $M > 0$ and $q \in \mathbb{N}$ such that

$$|\langle f(\mathbf{y}), \rho(\mathbf{y})\rangle| \leq M \sum_{|\mathbf{j}|=0}^{q} \left\|\mathbf{D}^{\mathbf{j}}\rho\right\|_{K,\infty} \quad \forall \rho \in \mathcal{E}(\mathbb{R}^n), \tag{3.10}$$

where $\|\rho\|_{K,\infty} = \sup\{|\rho(\mathbf{x})| : \mathbf{x} \in K\}$. There exist $r_0 > 0$ and constants $M_{\mathbf{j}} > 0$ such that $\left|\mathbf{D}^{\mathbf{j}}\phi(\mathbf{x})\right| \leq M_{\mathbf{j}} |\mathbf{x}|^{\beta - |\mathbf{j}|}$ for $|\mathbf{x}| \geq r_0$ and $|\mathbf{j}| \leq q$. Let L be a compact subset of $\mathbb{R}^n \setminus K$, and let $t_0 > 0$ be such that if $0 < t \leq t_0$ then $t^{-1}|\mathbf{x} - \mathbf{y}| \geq r_0$ for all $\mathbf{x} \in L$, $\mathbf{y} \in K$. Then, since

$$F(\mathbf{x}, t) = t^{-n} \left\langle f(\mathbf{y}), \phi\left(t^{-1}(\mathbf{y} - \mathbf{x})\right)\right\rangle, \tag{3.11}$$

it follows that for $0 < t \leq t_0$,

$$|F(\mathbf{x}, t)| \leq M_2 t^{-n-\beta}, \quad \forall \mathbf{x} \in L, \tag{3.12}$$

where $M_2 = M \sum_{|\mathbf{j}|=0}^{q} M_{\mathbf{j}}$ is a constant. Since $-\beta - n > 0$, we obtain that (3.9) holds uniformly on $\mathbf{x} \in L$. □

The definition of the Łojasiewicz point value is that if $f \in \mathcal{D}'(\mathbb{R}^n)$ then $f(\mathbf{x}_0) = \gamma$ distributionally if $\langle f(\mathbf{x}_0 + \varepsilon\mathbf{x}), \phi(\mathbf{x})\rangle \to \gamma \int_{\mathbb{R}^n} \phi(\mathbf{x})\, \mathrm{d}\mathbf{x}$, whenever $\phi \in \mathcal{D}(\mathbb{R}^n)$. If f belongs to a smaller class of distributions, then $\langle f(\mathbf{x}_0 + \varepsilon\mathbf{x}), \phi(\mathbf{x})\rangle$ will be defined for test functions of a larger class, not only for those of $\mathcal{D}(\mathbb{R}^n)$, and one may ask whether this remains true in that case. There are cases where it is not true, for instance if $f \in \mathcal{E}'(\mathbb{R})$ [3]. However, it was shown in [3] that in the one variable case, it holds if $f(x_0) = \gamma$ distributionally and the following conditions are satisfied:

$$f(x) = O(|x|^{\alpha}) \quad (\mathrm{C}), \text{ as } |x| \to \infty, \tag{3.13}$$

$$\phi(x) = O\left(|x|^{\beta}\right), \text{ strongly as } |x| \to \infty, \tag{3.14}$$

and $\alpha + \beta < -1$, $\beta < -1$. In particular, it is valid when $f \in \mathcal{S}'(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R})$ [3, 7, 9]. Actually a corresponding result is valid in several variables, and the proof is basically the same.

Theorem 2. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$ with $f(\mathbf{x}_0) = \gamma$ distributionally. Let $\phi \in \mathcal{E}(\mathbb{R}^n)$. Suppose that*

$$f(\mathbf{x}) = O(|\mathbf{x}|^{\alpha}) \quad (\mathrm{C}), \text{ as } |\mathbf{x}| \to \infty, \tag{3.15}$$

$$\phi(\mathbf{x}) = O\left(|\mathbf{x}|^{\beta}\right), \text{ strongly as } |\mathbf{x}| \to \infty, \tag{3.16}$$

$$\alpha + \beta < -n, \quad \text{and} \quad \beta < -n. \tag{3.17}$$

Then

$$\lim_{\varepsilon\to 0}\left\langle f\left(\mathbf{x}_0+\varepsilon\mathbf{x}\right),\phi\left(\mathbf{x}\right)\right\rangle=\gamma\int\limits_{\mathbb{R}^n}\phi\left(\mathbf{x}\right)\,\mathrm{d}\mathbf{x}\,. \tag{3.18}$$

Proof. Suppose that $\mathbf{x}_0=\mathbf{0}$. There exists a muti-index $\mathbf{k}$ and two primitives of f, $\mathbf{D}^{\mathbf{k}}G_1=\mathbf{D}^{\mathbf{k}}G_2=f$ such that they are continuous and

$$G_1\left(\mathbf{x}\right)=O\left(\left|\mathbf{x}\right|^{\alpha+|\mathbf{k}|}\right)\,,\quad \text{as }\ \left|\mathbf{x}\right|\to\infty\,, \tag{3.19}$$

$$G_2\left(\mathbf{x}\right)=\frac{\gamma\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!}+o\left(\left|\mathbf{x}\right|^{|\mathbf{k}|}\right)\,,\quad \text{as }\ \left|\mathbf{x}\right|\to 0\,. \tag{3.20}$$

Hence we can combine them into a single function G that satisfies

$$\begin{aligned}G(\mathbf{x})&=\frac{\gamma\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!}+o\left(\left|\mathbf{x}\right|^{|\mathbf{k}|}\right)\,,\quad \text{as }\ \left|\mathbf{x}\right|\to 0\,,\\ \left|G\left(\mathbf{x}\right)\right|&\le M\left|\mathbf{x}\right|^{|\mathbf{k}|}\,,\quad \text{for }\ \left|\mathbf{x}\right|\le 1\,,\\ \left|G\left(\mathbf{x}\right)\right|&\le M\left|\mathbf{x}\right|^{\alpha+|\mathbf{k}|}\,,\quad \text{for }\ \left|\mathbf{x}\right|\ge 1\,,\end{aligned}$$

and

$$f=g+\mathbf{D}^{\mathbf{k}}G\ , \tag{3.21}$$

where g has compact support and g vanishes near the origin. Then (3.18) holds for g (with $\gamma=0$), because of the Lemma 1. Therefore it is enough to prove (3.18) if $f=\mathbf{D}^{\mathbf{k}}G$; but in this case we may use the Lebesgue dominated convergence theorem to obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\left\langle f\left(\varepsilon\mathbf{x}\right),\phi\left(\mathbf{x}\right)\right\rangle&=\lim_{\varepsilon\to 0}\left(-1\right)^{|\mathbf{k}|}\varepsilon^{-|\mathbf{k}|}\int_{\mathbb{R}^n}G\left(\varepsilon\mathbf{x}\right)\mathbf{D}^{\mathbf{k}}\phi\left(\mathbf{x}\right)\,\mathrm{d}\mathbf{x}\\ &=\frac{\left(-1\right)^{|\mathbf{k}|}\gamma}{\mathbf{k}!}\int_{\mathbb{R}^n}\mathbf{x}^{\mathbf{k}}\mathbf{D}^{\mathbf{k}}\phi\left(\mathbf{x}\right)\,\mathrm{d}\mathbf{x}\\ &=\gamma\int_{\mathbb{R}^n}\phi\left(\mathbf{x}\right)\,\mathrm{d}\mathbf{x}\,,\end{aligned}$$

as required. □

In particular, (3.18) holds if $f\in\mathcal{S}'\left(\mathbb{R}^n\right)$ and $\phi\in\mathcal{S}\left(\mathbb{R}^n\right).$

Using the same argument as in the last proof we can prove that if $f\left(\mathbf{x}\right)=0$ for $\mathbf{x}\in\Omega$, an open set, and the conditions (3.15), (3.16), and (3.17) are satisfied, then the convergence in (3.18) is uniform on compacts of Ω.

We can now extend the distributional convergence of the $\phi-$transform, Theorem 1, to other cases.

Theorem 3. *If $\phi \in \mathcal{E}(\mathbb{R}^n)$ and $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfy the conditions (3.15), (3.16), and (3.17), then*

$$\lim_{t\to 0^+} F(\mathbf{x},t) = f(\mathbf{x}) \ , \tag{3.22}$$

distributionally in the space $\mathcal{D}'(\mathbb{R}^n)$, that is, if $\rho \in \mathcal{D}(\mathbb{R}^n)$, then

$$\lim_{t\to 0^+} \langle F(\mathbf{x},t), \rho(\mathbf{x})\rangle = \langle f(\mathbf{x}), \rho(\mathbf{x})\rangle \ . \tag{3.23}$$

In particular, distributional convergence, (3.22), holds if $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$.

Proof. We proceed as in the proof of the Theorem 1 by observing that $\langle F(\mathbf{x},t), \rho(\mathbf{x})\rangle = \langle \varrho(t\mathbf{y}), \phi(\mathbf{y})\rangle$, where $\varrho(\mathbf{z}) = \langle f(\mathbf{x}), \rho(\mathbf{x}-\mathbf{z})\rangle$. Next we observe that ϱ is a smooth function, and that it satisfies $\varrho(\mathbf{x}) = O(|\mathbf{x}|^{\alpha})$ (C), as $|\mathbf{x}| \to \infty$. Indeed, there exists a multi-index $\mathbf{k}$ and a primitive of f of that order, $\mathbf{D}^{\mathbf{k}}G = f$, which is an ordinary function for large arguments and satisfies $|G(\mathbf{x})| = O\left(|\mathbf{x}|^{|\mathbf{k}|+\alpha}\right)$ as $|\mathbf{x}| \to \infty$. We have then that

$$\begin{aligned}\varrho(\mathbf{z}) &= \left\langle \mathbf{D}^{\mathbf{k}}_{\mathbf{x}} G(\mathbf{x}), \rho(\mathbf{x}-\mathbf{z})\right\rangle \\ &= \mathbf{D}^{\mathbf{k}}_{\mathbf{z}} \langle G(\mathbf{x}), \rho(\mathbf{x}-\mathbf{z})\rangle \ ,\end{aligned}$$

and $\langle G(\mathbf{x}), \rho(\mathbf{x}-\mathbf{z})\rangle = \int_{\operatorname{supp}\rho} G(\mathbf{x}+\mathbf{z})\,\rho(\mathbf{x})\,\mathrm{d}\mathbf{x} = O\left(|\mathbf{z}|^{|\mathbf{k}|+\alpha}\right)$ as $|\mathbf{z}| \to \infty$, since $\operatorname{supp}\rho$ is compact. Hence, Theorem 2 allows us to obtain that $\lim_{t\to 0^+} \langle \varrho(t\mathbf{y}), \phi(\mathbf{y})\rangle = \varrho(\mathbf{0}) = \langle f(\mathbf{x}), \rho(\mathbf{x})\rangle$. □

Observe also if $\phi \in \mathcal{E}(\mathbb{R}^n)$ and $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfy the conditions (3.15), (3.16), and (3.17), then when the distributional point value $f(\mathbf{x}_0)$ exists, then $F(\mathbf{x},t) \to f(\mathbf{x}_0)$ as $(\mathbf{x},t) \to (\mathbf{x}_0, 0)$ in an angular fashion.

4. Measures and the $\phi-$transform

We shall use the following nomenclature. A (Radon) measure would mean a *positive* functional in the space of compactly supported continuous functions, which would be denoted by integral notation such as $\mathrm{d}\mu$, or by distributional notation, $f = f_\mu$, so that

$$\langle f, \phi\rangle = \int_{\mathbb{R}^n} \phi(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x}) \,, \tag{4.1}$$

and $\langle f, \phi\rangle \geq 0$ if $\phi \geq 0$. A signed measure is a real continuous functional in the space of compactly supported continuous functions, denoted as, say $\mathrm{d}\nu$, or as $g = g_\nu$. Observe that any signed measure can be written as $\nu = \nu_+ - \nu_-$, where $\nu_\pm$ are measures concentrated on disjoint sets. We shall also use the Lebesgue decomposition, according to which any signed measure ν can be written as $\nu = \nu_{\text{abs}} + \nu_{\text{sig}}$, where ν_{abs} is absolutely continuous with respect to the Lebesgue measure, so that it corresponds to a regular distribution, while ν_{sig} is a signed measure concentrated on a set of Lebesgue measure zero. We shall also need to consider the measures $(\nu_{\text{sig}})_{\pm} = (\nu_{\pm})_{\text{sig}}$, the positive and negative singular parts of ν.

Our first results are very simple, but useful.

Theorem 4. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Let U be an open set of $\mathbb{R}^n$. Then f is a measure in U if and only if its $\phi-$transform $F = F_\phi\{f\}$ with respect to a given normalized, positive test function $\phi \in \mathcal{D}(\mathbb{R}^n)$ satisfies $F(\mathbf{x}, t) \geq 0$ for all $(\mathbf{x}, t) \in \mathfrak{U}$, where $\mathfrak{U}$ is some open subset of $\mathbb{H}$ with $U \subset \overline{\mathfrak{U}} \cap \partial\mathbb{H}$.*

Proof. If f is a measure in U, and $\phi(\mathbf{x}) = 0$ for $|\mathbf{x}| \geq R$, then $F(\mathbf{x}, t) \geq 0$ if the ball of center $\mathbf{x}$ and radius Rt is contained in U, and the set of such points $(\mathbf{x}, t) \in \mathbb{H}$ could be taken as $\mathfrak{U}$. Conversely, if such $\mathfrak{U}$ exists then $\langle f, \psi \rangle = \lim_{t \to 0} \langle F(\mathbf{x}, t), \psi(\mathbf{x}) \rangle \geq 0$ whenever $\psi \in \mathcal{D}(\mathbb{R}^n)$, $\psi \geq 0$, and $\operatorname{supp} \psi \subset U$. □

Theorem 5. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Then f is a measure if and only if its $\phi-$transform $F = F_\phi\{f\}$ with respect to a given normalized, positive test function $\phi \in \mathcal{D}(\mathbb{R}^n)$ satisfies $F(\mathbf{x}, t) \geq 0$ for all $(\mathbf{x}, t) \in \mathbb{H}$.*

Proof. The proof is clear. □

If $\mathbf{x}_0 \in \mathbb{R}^n$ we shall denote by $C_{\mathbf{x}_0,\theta}$ the cone in $\mathbb{H}$ starting at $\mathbf{x}_0$ of angle $\theta \geq 0$,

$$C_{\mathbf{x}_0,\theta} = \{(\mathbf{x}, t) \in \mathbb{H} : |\mathbf{x} - \mathbf{x}_0| \leq (\tan\theta)t\} \ . \tag{4.2}$$

If $f \in \mathcal{D}'(\mathbb{R}^n)$ is real valued and $\mathbf{x}_0 \in \mathbb{R}^n$ then we consider the upper and lower angular values of its $\phi-$transform,

$$f^{+}_{\phi,\theta}(\mathbf{x}_0) = \limsup_{\substack{(\mathbf{x},t)\to(\mathbf{x}_0,0) \\ (\mathbf{x},t)\in C_{\mathbf{x}_0,\theta}}} F(\mathbf{x}, t) \ , \tag{4.3}$$

$$f^{-}_{\phi,\theta}(\mathbf{x}_0) = \liminf_{\substack{(\mathbf{x},t)\to(\mathbf{x}_0,0) \\ (\mathbf{x},t)\in C_{\mathbf{x}_0,\theta}}} F(\mathbf{x}, t) \ . \tag{4.4}$$

The quantities $f^{\pm}_{\phi,\theta}(\mathbf{x}_0)$ are well defined at all points $\mathbf{x}_0$, but, of course, they could be infinite.

Theorem 6. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Let U be an open set. Then f is a measure in U if and only if its $\phi-$transform $F = F_\phi\{f\}$ with respect to a given normalized, positive test function $\phi \in \mathcal{D}(\mathbb{R}^n)$ satisfies*

$$f^{-}_{\phi,\theta}(\mathbf{x}) \geq 0 \qquad \forall \mathbf{x} \in U\,,\ \forall \theta \in [0, \pi/2)\,. \tag{4.5}$$

Proof. If f is a measure in U, then $F \geq 0$ in some open set of $\mathbb{H}$, $\mathfrak{U}$ with $U \subset \overline{\mathfrak{U}} \cap \partial\mathbb{H}$, and thus (4.5) is satisfied. Conversely, let us show that if f is not a measure in U then (4.5) is not satisfied. First, if f is not a measure then there exists $\eta > 0$ such that $g = f + \eta$ is not a measure; let G be the $\phi-$transform of g. There exists an open ball B, with $\overline{B} \subset U$, such that g is not a measure in B. Using Theorem 4, if $0 < \varepsilon < 1$ we can find $(\mathbf{x}_1, t_1) \in \mathbb{H}$ with $\mathbf{x}_1 \in \overline{B}$ and $t_1 < \varepsilon$, such that $G(\mathbf{x}_1, t_1) < 0$.

The test function ϕ has compact support, so suppose that $\phi(\mathbf{x}) = 0$ for $|\mathbf{x}| \geq R$. Since $G(\mathbf{x}_1, t_1)$ depends only on the values of g on the closed ball $|\xi - \mathbf{x}_1| \leq Rt_1$, it follows that g is not a measure in that ball and consequently given $S > R$ and δ small enough, there exist t_δ and ξ_δ with $|\xi_\delta - \mathbf{x}_1| \leq St_1$ such that $G(\xi_\delta, t_\delta) < 0$. Let $0 < \alpha < 1$, and choose ε such that the distance from $\overline{B}$ to the complement of U is bigger than $S\varepsilon(1-\alpha)^{-1}$. Hence we can define recursively two sequences $\{\mathbf{x}_n\}$ and $\{t_n\}$ such that

$$|\mathbf{x}_n - \mathbf{x}_{n-1}| \leq St_{n-1}\,, \qquad 0 < t_n < \alpha t_{n-1}\,, \qquad G(\mathbf{x}_n, t_n) < 0\,. \tag{4.6}$$

The sequence $\{\mathbf{x}_n\}$ converges to some $\mathbf{x}^*$, because $\sum_{n=1}^{\infty} |\mathbf{x}_{n+1} - \mathbf{x}_n|$ converges, due to the inequality $|\mathbf{x}_{n+1} - \mathbf{x}_n| \leq S\alpha^{n-1}t_1$. Then $\mathbf{x}^* \in U$, since $|\mathbf{x}^* - \mathbf{x}_1| \leq S\varepsilon(1-\alpha)^{-1}$. Actually,

$$|\mathbf{x}^* - \mathbf{x}_n| \leq \sum_{k=n}^{\infty} |\mathbf{x}_{k+1} - \mathbf{x}_k| \leq \frac{St_n}{1-\alpha}\,, \tag{4.7}$$

and it also follows that $(\mathbf{x}_n, t_n) \in C_{\mathbf{x}^*,\theta}$ if $\tan\theta = S(1-\alpha)^{-1}$, and thus

$$g^{-}_{\phi,\theta}(\mathbf{x}^*) \leq 0\,. \tag{4.8}$$

But (4.8) in turn yields that $f^{-}_{\phi,\theta}(\mathbf{x}^*) < -\eta < 0$. □

If f is a signed measure then it has point values almost everywhere and thus the angular limit of its $\phi-$transform exists almost everywhere and equals the absolutely continuous part of the distribution. Therefore we immediately obtain the following result.

Theorem 7. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Suppose its $\phi-$transform $F = F_\phi\{f\}$ with respect to a given normalized, positive test function $\phi \in \mathcal{D}(\mathbb{R}^n)$*

satisfies

$$f_{\phi,\theta}^{-}(\mathbf{x}) \geq -M\,, \qquad \forall \mathbf{x} \in U\,,\ \forall \theta \in [0,\pi/2)\,, \tag{4.9}$$

where U is an open set and where M is a constant. Then the angular boundary limit

$$f_{\text{ang}}(\mathbf{x}) = \lim_{\substack{(\mathbf{x},t)\to(\mathbf{x}_0,0)\\ \text{angular}}} F(\mathbf{x},t)\ , \tag{4.10}$$

exists almost everywhere in U and defines a locally integrable function. Also there exists a singular measure μ_+ such that in U

$$f = f_{\text{ang}} + \mu_+\,. \tag{4.11}$$

Proof. Indeed, Theorem 6 yields that $f + M$ is a measure in U, whose Lebesgue decomposition yields (4.11), after a small rearrangement of terms. □

We also obtain the following result on the existence of almost everywhere angular limits of the $\phi-$transform.

Theorem 8. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Suppose its $\phi-$transform $F = F_\phi\{f\}$ with respect to a given normalized, positive test function $\phi \in \mathcal{D}(\mathbb{R}^n)$ satisfies*

$$M_+ \geq f_{\phi,\theta}^{+}(\mathbf{x}) \geq f_{\phi,\theta}^{-}(\mathbf{x}) \geq -M_-\,, \qquad \forall \mathbf{x} \in U\,,\ \forall \theta \in [0,\pi/2)\,. \tag{4.12}$$

where U is an open set and where $M_\pm$ are constants. Then the angular boundary limit

$$f_{\text{ang}}(\mathbf{x}) = \lim_{\substack{(\mathbf{x},t)\to(\mathbf{x}_0,0)\\ \text{angular}}} F(\mathbf{x},t)\ , \tag{4.13}$$

exists almost everywhere in U and defines a locally integrable function, and the distribution f is a regular distribution equal to f_{ang} in U :

$$\langle f(\mathbf{x}), \psi(\mathbf{x})\rangle = \int_{\mathbb{R}^n} f_{\text{ang}}(\mathbf{x})\,\psi(\mathbf{x})\ \mathrm{d}\mathbf{x}\,, \tag{4.14}$$

for all $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subset U$.

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