ON TAUBER'S SECOND TAUBERIAN THEOREM

RICARDO ESTRADA AND JASSON VINDAS

Abstract

We study Tauberian conditions for the existence of Cesàro limits in terms of the Laplace transform. We also analyze Tauberian theorems for the existence of distributional point values in terms of analytic representations. The development of these theorems is parallel to Tauber's second theorem on the converse of Abel's theorem. For Schwartz distributions, we obtain extensions of many classical Tauberians for Cesàro and Abel summability of functions and measures. We give general Tauberian conditions in order to guarantee (C, β) summability for a given order β. The results are directly applicable to series and Stieltjes integrals, and we therefore recover the classical cases and provide new Tauberians for the converse of Abel's theorem where the conclusion is Cesàro summability rather than convergence. We also apply our results to give new quick proofs of some theorems of Hardy-Littlewood and Szász for Dirichlet series.

1 Introduction

Tauberian Theory was initiated in 1897 by two simple theorems of Tauber for the converse of Abel's theorem [36] (see also [19, p. 11]). The present article is dedicated to providing extensions of Tauber's second theorem, in several directions.

Let us state Tauber's original theorems. Let us recall from [12, 19] that a sequence of complex numbers $\{c_n\}_{n=0}^{\infty}$ is said to be Abel summable to the number γ if $F(r) = \sum_{n=0}^{\infty} c_n r^n$ converges for $|r| < 1$ and $\lim_{r \to 1^-} F(r) = \gamma$. In such a case one writes

$$\sum_{n=0}^{\infty} c_n = \gamma \quad (\text{A}).$$

THEOREM 1.1 (Tauber's first theorem). *If $\sum_{n=0}^{\infty} c_n = \gamma$ (A) and*

$$(1) \qquad\qquad c_n = o\left(\frac{1}{n}\right), \quad n \to \infty,$$

then $\sum_{n=0}^{\infty} c_n$ converges to γ.

THEOREM 1.2 (Tauber's second theorem). *If $\sum_{n=0}^{\infty} c_n = \gamma$ (A) and*

$$(2) \qquad\qquad \sum_{n=1}^{N} n c_n = o(N), \quad N \to \infty,$$

then $\sum_{n=0}^{\infty} c_n$ converges to γ.

Tauber's theorems are very simple to prove [36, 12]. In 1910, Littlewood [20] gave his celebrated extension of Tauber's first theorem, where he substituted the Tauberian condition (1) by the weaker one $c_n = O\left(n^{-1}\right)$ and obtained the same conclusion of convergence as in Theorem 1.1. Actually, it can be shown that the hypotheses imply the (C, β) summability for any $\beta > -1$ [13]. It turns out that Littlewood's theorem is much deeper and difficult to prove than Theorem 1.1. Two years later, Hardy and Littlewood [13] conjectured that the condition $n c_n > -K$ would be enough to ensure the convergence; indeed, they provided a proof later in [14].

A version of Tauber's second theorem for Stieltjes integrals appeared in [47] (see [19, p. 28]).

Extensions of Theorem 1.2 are also known. It is natural to ask whether the replacement of (2) by a big O condition would lead to convergence; unfortunately, it does not suffice (see [30] for example). Nevertheless, one gets $(C, 1)$ summability as shown in the next theorem of Szász [34] (see also [29, 30, 35]), where even less is assumed.

THEOREM 1.3 (Szász, [34]). *Suppose that $\sum_{n=0}^{\infty} c_n = \gamma$ (A). Then the Tauberian condition*

$$(3) \qquad\qquad \sum_{n=1}^{N} n c_n > -KN,$$

for some $K > 0$, implies that $\sum_{n=0}^{\infty} c_n = \gamma$ (C, 1).

We will actually show (see Corollary 4.17 below) that if a two-sided condition is assumed instead of (3), then the series is summable (C, β) for all $\beta > 0$. It should be noticed that Theorem 1.3 includes Hardy-Littlewood's theorem quoted above (just apply Hardy's theorem for $(C, 1)$ summability [12, p. 121]). Versions of Theorem 1.3 for Dirichlet series can be found in [34] and [3, Sec. 3.8].

Tauberian theorems for power series have stimulated the creation of many interesting methods and theories in order to obtain extensions and easier proofs of them. Among the classical ones, one could mention those of Wiener [48] and Karamata [17, 18]. Other important ones come from the theory of generalized functions. In [44], Vladimirov obtained a multidimensional extension of Hardy-Littlewood type theorems for positive measures. Later on, the results from

[44] were generalized to include tempered distributions, resulting in a powerful multidimensional Tauberian theory for the Laplace transform [5, 46] (see also [6]). Distributional Abelian and Tauberian theorems for other integral transforms are investigated in [22, 25, 26]. Other related results are found in [23, 24, 27]. Some Tauberian results for distributions have interesting consequences in the theory of Fourier series [11].

Recently, the authors were able to deduce Littlewood's Tauberian theorem [20] from a Tauberian theorem for distributional point values [41]; actually, the method recovered the more general version for Dirichlet series proved first by Ananda Rau [1]. A similar approach, but with a more comprehensive character, will be taken in this paper. In Section 2, we collect some known results and explain the notation to be used in the course of the article. Section 3 is devoted to the study of Cesàro limits and summability in the context of Schwartz distributions; we extend the known definitions in order to consider fractional orders of summability, then we provide several technical Tauberian theorems which will establish the link between results for generalized functions and Stieltjes integrals. The main part of the article is Section 4. There, we first show a theorem for distributional point values which generalizes Theorem 1.3. Moreover, our theorem is capable to recover Theorem 1.3, and it is applicable to much more situations. Finally, Section 5 is dedicated to applications of the distributional method, we generalize [35, Thm. B] from series to Stieltjes integrals, and we also give quick proofs of some classical Tauberians of Hardy-Littlewood [15] and Szász [32, 33, 34] for Dirichlet series.

2   Preliminaries and notation

We will assume the reader is familiar with the basic notions from the theory of summability of numerical series and Stieltjes integrals, specifically with the summability methods by Cesàro, Riesz and Abel means. There is a very rich and extensive literature on the subject; for instance, we refer to [3, 12, 19].

The spaces of test functions and distributions $\mathcal{D}(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$, $\mathcal{D}'(\mathbb{R})$, and $\mathcal{S}'(\mathbb{R})$ are well known for most analysts. The space of all C^∞-functions over the real line with its canonical topology is denoted by $\mathcal{E}(\mathbb{R})$; hence, its dual, $\mathcal{E}'(\mathbb{R})$, is the space of distributions with compact support. We refer to [31, 45] for properties of these spaces.

A distribution $f \in \mathcal{D}'(\mathbb{R})$ is said to have a distributional point value (in the sense of Łojasiewicz [21]) at $x = x_0 \in \mathbb{R}$ if $\lim_{\varepsilon \to 0} f(x_0 + \varepsilon x) = \gamma$ in the weak topology of $\mathcal{D}'(\mathbb{R})$; that is, for each $\phi \in \mathcal{D}(\mathbb{R})$,

$$(4) \qquad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\langle f(x), \phi\left(\frac{x - x_0}{\varepsilon}\right) \right\rangle = \gamma \int_{-\infty}^{\infty} \phi(x)\mathrm{d}x \ .$$

In this case we write $f(x_0) = \gamma$, *distributionally.* Suppose that $f \in \mathcal{S}'(\mathbb{R})$, it is then natural to ask whether (4) would hold for $\phi \in \mathcal{S}(\mathbb{R})$; the answer is positive, as shown in [9, 43].

More generally, one can consider asymptotic relations

$$(5) \qquad f(x_0 + \varepsilon x) \sim \varepsilon^\alpha g(x) \quad \text{as } \varepsilon \to 0^+ \ ,$$

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in the weak topology of $\mathcal{D}'(\mathbb{R})$ or $\mathcal{S}'(\mathbb{R})$. These relations are then particular cases of the so called *quasiasymptotic* behavior of distributions at finite points [10, 27, 43, 46]. The distribution g in (5) must necessarily be homogeneous of degree α [26, 27]. We also have that for tempered distributions the asymptotic relation (5) in $\mathcal{D}'(\mathbb{R})$ implies the same asymptotic behavior in $\mathcal{S}'(\mathbb{R})$ [9, 43]. The same is true for the relations $f(x_0 + \varepsilon x) = O(\varepsilon^\alpha)$ as $\varepsilon \to 0^+$ [9, 39]. Observe also that (5) is a local property, thus, it still makes sense if the distribution f is only defined in a neighborhood of $x = x_0$.

We say that $f \in \mathcal{D}'(\mathbb{R})$ is *distributionally bounded* at $x = x_0 \in \mathbb{R}$ if $f(x_0 + \varepsilon x) = O(1)$ as $\varepsilon \to 0^+$ in $\mathcal{D}'(\mathbb{R})$.

These notions have their obvious analogs at infinity [10, 27, 37, 39, 46]; indeed, one considers the behavior of $f(\lambda x)$ as $\lambda \to \infty$ in the weak topology of a distribution space.

Let $f \in \mathcal{D}'(\mathbb{R})$. A function $U(z)$, harmonic for $\Im m\, z > 0$, is said to be a harmonic representation of f if $\lim_{y \to 0^+} U(x + iy) = f(x)$ in $\mathcal{D}'(\mathbb{R})$ (see [2]). In such a case we write $f(x) = U(x + i0)$. If $f(x) = U(x + i0)$ and $f(x_0) = \gamma$, distributionally, then $U(z) \to \gamma$ as $z \to x_0$, in an angular (or nontangential) fashion [4, 9, 27, 39]; however, the converse is not true in general [8]. In the case of boundary values of analytic functions, we have the following *Tauberian* theorem [41].

THEOREM 2.1. *Let F be analytic in a rectangular region of the form $(a, b) \times (0, R)$. Suppose $f(x) = F(x + i0)$ in $\mathcal{D}'(a, b)$. Let $x_0 \in (a, b)$ such that $F(x_0 + iy) \to \gamma$ as $y \to 0^+$. The Tauberian condition "f is distributionally bounded at $x = x_0$" implies that $f(x_0) = \gamma$, distributionally.*

We shall employ several special distributions in this article. We will follow the notation from [10]. The Dirac delta distribution is denoted by δ, as usual. H is the Heaviside function, i.e., the characteristic function of $(0, \infty)$. Let Γ be the Euler gamma function. The analytic family of distributions $x_+^{\beta-1}/\Gamma(\beta)$ is given by

$$\left\langle \frac{t_+^{\beta-1}}{\Gamma(\beta)}, \phi(t) \right\rangle = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1}\phi(t)\mathrm{d}t \ ,$$

whenever $\Re e\, \beta > 0$, and they are obtained by analytic continuation of the last equation in the other cases [10, p. 65]. In particular, we have $(x_+^{\beta-1}/\Gamma(\beta))\,|_{\beta=0} = \delta(x)$, the Dirac delta distribution.

We will work with the Fourier transform

$$\hat{\phi}(x) = \int_{-\infty}^\infty \phi(t)e^{-ixt}\mathrm{d}t \ ,$$

defined for $\phi \in \mathcal{S}(\mathbb{R})$, and by duality on $\mathcal{S}'(\mathbb{R})$.

Let s be a function of local bounded variation. Furthermore, suppose that s has support in $[0, \infty)$. We shall always *assume* that s is *normalized* at $x = 0$, in the sense that $s(0) = 0$, so that the distributional derivative, $s' \in \mathcal{D}'(\mathbb{R})$, of s is given by the Stieltjes integral

$$\langle s'(t), \phi(t) \rangle = \int_0^\infty \phi(t)\mathrm{d}s(t) \ .$$

Let μ be a Radon measure, that is, a continuous linear functional over the space of continuous functions with compact support. We can always associate to it a function of local bounded variation, which we shall denote by s_μ, so that $\mu = \mathrm{d}s_\mu$, i.e., $\langle \mu(t), \phi(t) \rangle = \int_{-\infty}^{\infty} \phi(t) \mathrm{d}s_\mu(t)$. If $|\mu|$ is the variation measure associated to μ, we also denote it by $|\mathrm{d}s_\mu|$.

3 Tauberian theorems for (C) summability

In this section we show Tauberian theorems for (C) summability of distributions and measures related to Theorems 1.2 and 1.3. We first study Cesàro limits and boundedness, of fractional orders, for distributions; then these notions are extended to distributional evaluations, which include as particular cases those of series and Stieltjes integrals. Next, a convexity theorem is shown. Finally, we present the Tauberian theorems.

3.1 Cesàro limits and boundedness.

Suppose that s is locally bounded on $[0, \infty)$. Recall [12, 19] that we write $\lim_{x \to \infty} s(x) = \gamma$ (C, β), $\beta > 0$, if

$$(6) \qquad \lim_{x \to \infty} \frac{\beta}{x} \int_0^x s(t) \left(1 - \frac{t}{x} \right)^{\beta - 1} \mathrm{d}t = \gamma \ .$$

When s is of local bounded variation, we may also denote this relation by $\int_0^\infty \mathrm{d}s(t) = \gamma$ (C, β), and the limit in the above equation might be replaced by the limit of $\int_0^x (1 - (t/x))^\beta \mathrm{d}s(t)$, as integration by parts shows (here we use the assumption $s(0) = 0$). If $1 \le \beta$, the notion (6) makes also sense for s being merely locally integrable.

The Cesàro behavior can also be defined for distributions [7, 10]. It includes the case of classical functions and Stieltjes integrals. We adopt in this paper new definitions, our purpose is to include fractional orders for Cesàro limits and boundedness of distributions at infinity; a similar approach has been followed in [42]. In the case of integral orders, it coincides with the definition from [7]. We will also define one-sided boundedness.

Given $f \in \mathcal{D}'(\mathbb{R})$, with support bounded at the left, its β-*primitive* is given by the convolution [45, p. 72]

$$f^{(-\beta)} = f * \frac{x_+^{\beta - 1}}{\Gamma(\beta)} \ .$$

DEFINITION 3.1. Let $f \in \mathcal{D}'(\mathbb{R})$, and $\beta \ge 0$. We say that f is bounded at infinity in the Cesàro sense of order β (in the (C, β) sense), and write

$$f(x) = O(1) \quad (\mathrm{C}, \beta), \quad \text{as } x \to \infty \ ,$$

if for a decomposition $f = f_- + f_+$ as sum of two distributions with the supports bounded on the right and left, respectively, one has that the β-primitive of f_+ is an ordinary function (locally integrable) for large arguments and satisfies the ordinary order relation

$$f_+^{(-\beta)}(x) = O\left(x^\beta \right), \quad x \to \infty \ .$$

A similar definition applies for the little o-symbol. We denote

$$\lim_{x \to \infty} f(x) = \gamma \quad (C, \beta)$$

if

$$f(x) = \gamma + o(1) \quad (C, \beta), \quad \text{as } x \to \infty,$$

that is, $f_+^{(-\beta)}$ is locally integral for large arguments and

$$f_+^{(-\beta)}(x) \sim \frac{\gamma x_+^{\beta}}{\Gamma(\beta + 1)}, \quad x \to \infty.$$

in the ordinary sense.

Naturally, there is some consistency to be checked in Definition 3.1. We show in the next proposition that Definition 3.1 is independent of the choice of f_+.

PROPOSITION 3.2. *Suppose that f has compact support. If $\beta \geq 0$ and $\alpha > -1$, then $f^{(-\beta)}(x) = o(x^{\beta+\alpha})$, $x \to \infty$. In particular,*

$$\lim_{x \to \infty} f(x) = 0 \quad (C, \beta), \quad \text{for each } \beta \geq 0.$$

PROOF. If β is a non-negative integer, the conclusion is obvious. Assume $\beta > 0$ is not a positive integer. We show that $f^{(-\beta)}$ is locally integrable for large arguments and $f^{-\beta}(x) = o(x^{\beta+\alpha})$, $x \to \infty$. Let k be a positive integer such that $f^{(-k)}$ is continuous over the whole real line. Then $f^{(-k)} = P + F$, where $P(x) = \sum_{j=0}^{k-1} a_j(x_+^j/j!)$, for some constants, and F is continuous on a certain compact interval $[a, b]$, and $F(x) = 0$ for $x \notin (a, b)$. We have that $f = P^{(k)} + F^{(k)}$. Note first that

$$P^{(k)} * \frac{x^{\beta-1}}{\Gamma(\beta)} = \sum_{j=0}^{k-1} a_j \delta^{(k-1-j)} * \frac{x_+^{(\beta-1)}}{\Gamma(\beta)} = \sum_{j=0}^{k-1} a_j \frac{x_+^{(\beta+j-k)}}{\Gamma(\beta + 1 + j - k)}$$

$$= O\left(x^{\beta-1}\right) = o\left(x^{\beta+\alpha}\right), \quad x \to \infty.$$

So, it is enough to show that

$$F^{(k)} * \left(x^{\beta-1}/\Gamma(\beta)\right) = F * \left(x^{\beta-k-1}/\Gamma(\beta-k)\right)$$

is locally integrable for large arguments and satisfies an order estimate $o(x^{\beta+\alpha})$ as $x \to \infty$. Indeed, we show it is locally integrable on $(b+1, \infty)$. If $\phi \in \mathcal{D}(\mathbb{R})$ is so that $\operatorname{supp} \phi \subseteq (b+1, \infty)$, then $\operatorname{supp} \phi * F(-t) \subseteq [1, \infty)$, hence,

$$\left\langle F * x^{\beta-k-1}, \phi(x) \right\rangle = \left\langle x^{\beta-k-1}, (F(-t) * \phi)(x) \right\rangle$$

$$= \int_1^{\infty} x^{\beta-k-1} \left(\int_{-\infty}^{\infty} F(t-x)\phi(t)\mathrm{d}t \right) \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \left(\int_1^{\infty} t^{\beta-k-1} F(x-t)\mathrm{d}t \right) \phi(x)\mathrm{d}x.$$

On the other hand if $x > b + 1$, we obtain, as $x \to \infty$,

$$\int_1^\infty t^{\beta-k-1} F(x-t) \mathrm{d}t = \int_a^b (x-t)^{\beta-k-1} F(t) \mathrm{d}t = O\left(x^{\beta-1-k}\right) = o\left(x^{\beta+\alpha}\right) \ .$$

$\square$

We now define one-sided boundedness. Recall that a positive distribution is nothing but a positive Radon measure.

DEFINITION 3.3. Let $f \in \mathcal{D}'(\mathbb{R})$, $\beta \geq 0$, and $\alpha > -1$. We say that f is bounded from below (or left bounded) near infinity by $O_L(x^\alpha)$ in the Cesàro sense of order β, and write

$$f(x) = O_L(x^\alpha) \quad (\mathrm{C},\beta), \quad \text{as } x \to \infty \ ,$$

if there exist a decomposition $f = f_- + f_+$, as sum of two distributions with the supports bounded on the right and left, respectively, a constant $K > 0$, and an interval (a, ∞) such that $\Re e\, f_+^{(-\beta)} + K x_+^{\alpha+\beta}$ and $\Im m\, f_+^{(-\beta)} + K x_+^{\alpha+\beta}$ are positive distributions on (a, ∞). A similar definition applies for right boundedness, in such a case we employ the symbol $O_R(x^\alpha)$.

In case f is one-sided bounded by $O_L(1)$ or $O_R(1)$, we simply say that f is one-sided bounded. Our definitions of Cesàro behavior have the following expected property.

PROPOSITION 3.4. *If f is Cesàro bounded (resp. has Cesàro limit, or is one-sided bounded) at infinity of order β, then it is Cesàro bounded (resp., has Cesàro limit, or is one-sided bounded) at infinity of order $\tilde\beta > \beta$.*

PROOF. It follows immediately from Proposition 3.2. $\square$

Observe that, because of Proposition 3.2, we can always assume in Definitions 3.1 and 3.3 that $f = f_+$, if needed. When we do not want to make reference to the order β in (C, β), we write (C). Most statements below are about complex-valued distributions. We will often drop $x \to \infty$ from the notation. Note that if $f(x) = O(1)$, $x \to \infty$, then $f_+ \in \mathcal{S}'(\mathbb{R})$ (here $f = f_- + f_+$ as in Definitions 3.1 and 3.3). In addition, it should be noticed that if both conditions $f(x) = O_L(1)$ and $f(x) = O_R(1)$ hold in the (C, β) sense, then $f(x) = O(1)$ (C, β).

3.2 Cesàro summability of distributional evaluations and the connection with series and Stieltjes integrals.

It is convenient for future purposes to use the notion of Cesàro summability of distributional evaluations [7, 10]. It will provide the link between our theorems for distributions and results for series and Stieltjes integrals. Let g be a distribution with the support bounded at the left and let $\phi \in \mathcal{E}(\mathbb{R})$; we say that the *distributional evaluation* $\langle g(x), \phi(x) \rangle$ exists and is equal to a number γ in the Cesàro sense (of order $\beta \geq 0$), and write

$$\langle g(x), \phi(x) \rangle = \gamma \quad (\mathrm{C}, \beta) \ ,$$

if the primitive G of ϕg with the support bounded at the left, i.e., $G = (\phi g)^{(-1)}$, satisfies $\lim_{x \to \infty} G(x) = \gamma$ (C,β). Note that when s is of local bounded variation with $s(x) = 0$ for $x \le 0$, we have

$$\int_0^\infty \phi(t)\mathrm{d}s(t) = \gamma \quad (\mathrm{C},\beta)$$

if and only if

$$\langle s'(t), \phi(t) \rangle = \gamma \quad (\mathrm{C},\beta) \ ;$$

and it explicitly means that

$$\lim_{x \to \infty} \int_0^x \phi(t) \left(1 - \frac{t}{x} \right)^\beta \mathrm{d}s(t) = \gamma \ .$$

A similar consideration applies for Cesàro boundedness. In particular, when $s(x) = \sum_{\lambda_n < x} c_n$, where $\lambda_n \nearrow \infty$, we have

$$\sum_{n=0}^\infty c_n = \gamma \quad (\mathrm{R}, \{\lambda_n\}, \beta)$$

if and only if

$$\left\langle \sum_{n=0}^\infty c_n \delta \left(x - \lambda_n \right), 1 \right\rangle = \gamma \quad (\mathrm{C},\beta) \ ,$$

where here (R) stands for the Riesz method by typical means [3, 10, 12].

Using the well-known equivalence between summability by $(\mathrm{R}, \{n\})$ means and Cesàro means [12, 16], we have that if $\beta \ge 0$, then $\sum_{n=0}^\infty c_n = \gamma$ (C,β) if and only if

$$\left\langle \sum_{n=0}^\infty c_n \delta \left(x - n \right), 1 \right\rangle = \gamma \quad (\mathrm{C},\beta) \ .$$

Finally, we will need the following observation in the future. Given a sequence $\{b_n\}_{n=0}^\infty$ and $\beta > 0$, write

$$b_N = O(N) \quad (\mathrm{C},\beta) \ ,$$

if the Cesàro means of order β of the sequence (not to be confused with the Cesàro means of a series) are $O(N)$, that is,

$$\sum_{n=0}^N \binom{N-n+\beta-1}{N-1} b_n = O(N^{\beta+1}) \ .$$

Likewise, we define the symbols O_R and O_L in the Cesàro sense for sequences. Following Ingham's method [16], we obtain the following useful equivalence.

LEMMA 3.5. *Let $\beta \ge 0$. The conditions*

(7) $$\sum_{n=0}^N c_n = O(N) \quad (\mathrm{C},\beta)$$

and

$$(8) \qquad \sum_{n=0}^{\infty} c_n \delta(x-n) = O(1) \quad (\mathrm{C}, \beta+1) \ , \qquad \text{as } x \to \infty \ ,$$

are equivalent. The same holds for the symbols O_R and O_L.

PROOF. Repeating the arguments from [16], with the obvious modifications, one is led to the equivalence between (7) and the relation

$$\sum_{n<x} (x-n)^{\beta} c_n = O(x^{\beta+1}), \quad (\text{resp. } O_R \text{ and } O_L),$$

which turns out to be the meaning of (8). $\qquad \square$

3.3 A convexity (Tauberian) theorem.

We now show a convexity theorem for the Cesàro limits of distributions. It generalizes [12, Thm. 70].

THEOREM 3.6. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that $\lim_{x\to\infty} f(x) = \gamma$ (C, β_2), for some $\beta_2 > 0$. If $f(x) = O_L(1)$ (C, β_1), then $\lim_{x\to\infty} f(x) = \gamma$ (C, β) for any $\beta \geq \beta_1 + 1$. The same conclusion holds if we replace $O_L(1)$ by $O_R(1)$. If now $f(x) = O(1)$ (C, β_1), as $x \to \infty$, then $\lim_{x\to\infty} f(x) = \gamma$ (C, β) for any $\beta > \beta_1$.*

Theorem 3.6 follows immediately from the next proposition. For the first part we give a proof based on the distributional ideas of Drozhzhinov and Zavialov [5, Lem. 3]; it may also be deduced from the classical results on asymptotics of derivatives ([19, pp. 34–37], [12, Thm. 112]). We give a direct proof of the second part.

PROPOSITION 3.7. *Let μ be a (real-valued) Radon measure supported in $[0, \infty)$ and $\alpha > -1$. Suppose that for some $\beta_1 > 1$,*

$$(9) \qquad \int_0^x (x-t)^{\beta_1-1} \mathrm{d}s_\mu(t) \sim \frac{\gamma \Gamma(\beta_1) \Gamma(\alpha+1)}{\Gamma(\beta_1+\alpha+1)} x^{\alpha+\beta_1} \ , \qquad x \to \infty \ .$$

If the one-sided condition $\mu = O_L(x^\alpha)$ is satisfied, i.e., in a neighborhood of infinity

$$C x_+^\alpha + \mu \quad \text{is a positive measure,}$$

for some constant C, then for any $\beta \geq 1$,

$$(10) \qquad \int_0^x \left(1 - \frac{t}{x}\right)^{\beta-1} \mathrm{d}s_\mu(t) \sim \frac{\gamma \Gamma(\beta) \Gamma(\alpha+1)}{\Gamma(\beta+\alpha+1)} x^{\alpha+1} \ , \qquad x \to \infty \ .$$

If in addition μ is absolutely continuous with respect to the Lebesgue measure, and the two-sided condition

$$F_\mu(x) = O(x^\alpha) \ , \qquad x \to \infty \ ,$$

is satisfied, where $F_\mu \in L^1_{\mathrm{loc}}(\mathbb{R})$ is such that $\mathrm{d}s_\mu(t) = F_\mu(t)\mathrm{d}t$, then (10) holds whenever $\beta > \max\{-\alpha, 0\}$.

PROOF. Let us show the first part of the theorem. By adding Cx_+^α to μ, we may assume $C = 0$, and so we are assuming that μ is a positive measure. Next (9) directly implies that μ has the quasiasymptotic behavior

$$(11) \qquad \mu(\lambda x) = \gamma \lambda^\alpha x_+^\alpha + o\left(\lambda^\alpha\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}),$$

[5, 37, 46], that is, for each test function $\phi \in \mathcal{D}(\mathbb{R})$,

$$\langle \mu(\lambda x), \phi(x) \rangle = \frac{1}{\lambda} \int_0^\infty \phi\left(\frac{x}{\lambda}\right) \mathrm{d}s_\mu(x) \sim \gamma \lambda^\alpha \int_0^\infty \phi(x) x^\alpha \mathrm{d}x .$$

But it is well known that for positive measures the quasiasymptotic behavior (11) is equivalent to the asymptotics of the primitive, i.e., (10) for $\beta = 1$,

$$s_\mu(\lambda) = \int_0^\lambda \mathrm{d}s_\mu(t) \sim \gamma \frac{\lambda^{\alpha+1}}{\alpha+1} , \quad \lambda \to \infty ,$$

from which (10) follows for any $\beta \geq 1$. This completes the proof of the first part.

For the second part, write $F := F_\mu$ and $s(x) := s_\mu(x) = \int_0^x F(t)\mathrm{d}t$. We assume that $|F(x)| \leq Mx^\alpha$ for some constant M and x large enough. Moreover, by Proposition 3.2, it is clear that we can assume this condition to hold for all x. We obtain from the first part that $s(x) \sim \gamma x^{\alpha+1}/(\alpha+1)$, $x \to \infty$. We also have that if $0 < r < 1$,

$$|s(rx) - s(x)| \leq M \int_{rx}^x t^\alpha \mathrm{d}t = \frac{M}{\alpha+1}(1-r)^{\alpha+1} x^{\alpha+1} .$$

Hence, if $\max\{-\alpha, 0\} < \beta < 1$,

$$\int_0^x F(t)\left(1 - \frac{t}{x}\right)^{\beta-1} \mathrm{d}t = \lim_{r \to 1^-} \int_0^{rx} F(t)\left(1 - \frac{t}{x}\right)^{\beta-1} \mathrm{d}t$$

$$= \lim_{r \to 1^-} \left(s(rx)(1-r)^{\beta-1} + \frac{\beta-1}{x} \int_0^{rx} s(t)\left(1 - \frac{t}{x}\right)^{\beta-2} \mathrm{d}t \right)$$

$$= \lim_{r \to 1^-} \left((1-r)^{\alpha+\beta} \frac{s(rx) - s(x)}{(1-r)^{\alpha+1}} + s(x)(1-r)^{\beta-1} + \frac{\beta-1}{x} \int_0^{rx} s(t)\left(1 - \frac{t}{x}\right)^{\beta-2} \mathrm{d}t \right)$$

$$= s(x) + \lim_{r \to 1^-} \frac{\beta-1}{x} \int_0^{rx} (s(t) - s(x))\left(1 - \frac{t}{x}\right)^{\beta-2} \mathrm{d}t$$

$$= (\beta-1) \int_0^1 (s(xt) - s(x))(1-t)^{\beta-2}\mathrm{d}t + \gamma \frac{x^{\alpha+1}}{\alpha+1} + o(x^{\alpha+1})$$

$$= x^{\alpha+1}\left((\beta-1) \int_0^1 \frac{s(xt) - s(x)}{x^{\alpha+1}(1-t)}(1-t)^{\beta-1}\mathrm{d}t + \frac{\gamma}{\alpha+1} + o(1) \right)$$

$$= \gamma \frac{\beta-1}{\alpha+1} x^{\alpha+1}\left(\int_0^1 (t^{\alpha+1} - 1)(1-t)^{\beta-2} + \frac{1}{\beta-1} + o(1) \right)$$

$$= \gamma \frac{\beta-1}{\alpha+1} x^{\alpha+1}\left(\frac{\Gamma(\beta-1)\Gamma(\alpha+2)}{\Gamma(\beta+\alpha+1)} - \frac{1}{\beta-1} + \frac{1}{\beta-1} + o(1) \right)$$

$$= \gamma \frac{\Gamma(\beta)\Gamma(\alpha+1)}{\Gamma(\beta+\alpha+1)} x^{\alpha+1} + o\left(x^{\alpha+1}\right) , \quad x \to \infty .$$

$\square$

3.4 Tauberian theorems for (C) summability.

We now analyze Tauberian conditions of Tauber's second type. For that, we need the following formula. Given $g \in \mathcal{S}'(\mathbb{R})$, with the support bounded at the left, its Laplace transform is denoted by $\mathcal{L}\{g;z\} := \langle g(t), e^{-zt}\rangle$, for $\Re e\, z > 0$.

LEMMA 3.8. *Suppose that $f \in \mathcal{D}'(\mathbb{R})$ has the support bounded at the left. Then*

(12) $$(xf)^{(-\beta)} = xf^{(-\beta)} - \beta f^{(-\beta-1)} \ .$$

PROOF. We first assume that $f \in \mathcal{S}'(\mathbb{R})$. We make use of the injectivity of the Laplace transform. Set $F(z) = \mathcal{L}\{f(t);z\}$. Then,

$$\mathcal{L}\left\{tf^{(-\beta)}(t);z\right\} = -\frac{d}{dz}\left(\mathcal{L}\left\{f^{(-\beta)}(t);z\right\}\right) = -\frac{d}{dz}\left(F(z)\mathcal{L}\left\{\frac{t_+^{(\beta-1)}}{\Gamma(\beta)};z\right\}\right)$$

$$= \beta\frac{F(z)}{z^{\beta+1}} - \frac{F'(z)}{z^\beta} = \mathcal{L}\left\{\beta f^{(-\beta-1)} + (tf)^{(-\beta)};z\right\} \ ,$$

which shows (12). In the general case we take a sequence $\{f_n\}_{n=0}^\infty$, with each f_n being tempered and having support on some fixed interval $[a,\infty)$, such that $f_n \to f$ in $\mathcal{D}'(\mathbb{R})$; then (12) is satisfied for each f_n. Thus, the continuity of the fractional integration operator [45], on $\mathcal{D}'[a,\infty)$, shows (12) for f, after passing to the limit. $\square$

We now connect Tauber's second type conditions with Cesàro boundedness.

LEMMA 3.9. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that $f(x) = O(1)$ (C, β_2), as $x \to \infty$, for some $\beta_2 \geq 0$. Then, the condition*

(i) *$xf'(x) = O(1)$ (C, $\beta_1 + 1$) holds if and only if $f(x) = O(1)$ (C, β_1),*

(ii) *$xf'(x) = O_L(1)$ (C, $\beta_1 + 1$) holds if and only if $f(x) = O_L(1)$ (C, β_1). The same is true if $O_L(1)$ is replaced by $O_R(1)$.*

PROOF. We show (i). Assume that f has the support bounded on the left. We can assume that β_2 has the form $\beta_2 = \beta_1 + k$, for some $k \in \mathbb{N}$. Let $g = xf'$, then, by Lemma 3.8,

$$xf^{(-\beta_1-k+1)}(x) = \beta_2 f^{(-\beta_1-k)}(x) + g^{(-\beta_1-k)}(x) = g^{(-\beta_1-k)}(x) + O\left(x^{\beta_1+k}\right) \ , \quad x \to \infty \ ;$$

then $f(x) = O(1)$ (C, $\beta_1 + k - 1$) if and only if $g(x) = O(1)$ (C, $\beta_1 + k$), as $x \to \infty$. A recursive argument shows that $f(x) = O(1)$ (C, β_1) if and only if $g(x) = O(1)$ (C, $\beta_1 + 1$).

The same proof applies for one-sided boundedness. $\square$

So, we immediately obtain the following result from Theorem 3.6.

THEOREM 3.10. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that $\lim_{x\to\infty} f(x) = \gamma$ (C, β_2) for some $\beta_2 \geq 0$. The Tauberian condition $xf'(x) = O(1)$ (C, $\beta_1 + 1$), for some $\beta_1 \geq 0$, implies that $\lim_{x\to\infty} f(x) = \gamma$ (C, β) for all $\beta > \beta_1$.*

PROOF. Indeed, we obtain, by Lemma 3.9, $f(x) = O(1)$ (C, β_1), as $x \to \infty$; hence, an application of Theorem 3.6 gives the result. $\square$

THEOREM 3.11. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that $\lim_{x\to\infty} f(x) = \gamma$ (C, β_2) for some $\beta_2 \geq 0$. The Tauberian condition $xf'(x) = O_L(1)$ $(C, \beta_1 + 1)$, for some $\beta_1 \geq 0$, implies that $\lim_{x\to\infty} f(x) = \gamma$ (C, β) for all $\beta \geq \beta_1 + 1$. The same holds if we replace $O_L(1)$ by $O_R(1)$.*

PROOF. From Lemma 3.9, we have $f(x) = O_L(1)$ (C, β_1), as $x \to \infty$; hence, again, we can an apply Theorem 3.6. $\square$

We also analyze a little o condition. It generalizes [12, Thm. 65] to distributions.

THEOREM 3.12. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that $\lim_{x\to\infty} f(x) = \gamma$ (C, β_2). If $\beta_2 > \beta_1 \geq 0$, a necessary and sufficient condition for the limit to hold (C, β_1) is $xf'(x) = o(1)$ $(C, \beta_1 + 1)$.*

PROOF. We retain the notation from the proof of Lemma 3.9. If $\lim_{x\to\infty} f(x) = \gamma$ $(C, k+\beta_1)$, for some $k > 0$, then the relation

$$xf^{(-\beta_1 - k + 1)}(x) = g^{(-\beta_1 - k)}(x) + (\beta_1 + k)f^{(-\beta_1 - k)}(x)$$
$$= g^{(-\beta_1 - k)}(x) + \frac{\gamma x^{\beta_1 + k}}{\Gamma(\beta_1 + k)} + o\left(x^{\beta_1 + k}\right)$$

shows the equivalence at level $k - 1$. A recursive argument proves that the equivalence should hold for $k = 1$. $\square$

We may state our results in terms of (C) summability of distributional evaluations. We obtain the next series of corollaries directly from Theorems 3.10, 3.11, and 3.12.

COROLLARY 3.13. *Let $g \in \mathcal{D}'(\mathbb{R})$ and $\phi \in \mathcal{E}(\mathbb{R})$. Suppose that $\operatorname{supp} g$ is bounded at the left and*

$$(13) \qquad \langle g(x), \phi(x) \rangle = \gamma \quad (C) \ .$$

(i) *If $x\phi(x)g(x) = O(1)$ $(C, \beta_1 + 1)$, as $x \to \infty$, for $\beta_1 \geq 0$, then*

$$\langle g(x), \phi(x) \rangle = \gamma \quad (C, \beta) \ , \quad for\ all\ \beta > \beta_1 \ .$$

(ii) *If $x\phi(x)g(x) = O_L(1)$ $(C, \beta_1 + 1)$, as $x \to \infty$, for $\beta_1 \geq 0$, then*

$$\langle g(x), \phi(x) \rangle = \gamma \quad (C, \beta) \ , \quad for\ all\ \beta \geq \beta_1 + 1 \ .$$

(iii) *Given $\beta \geq 0$, a necessary and sufficient condition for (13) to hold (C, β) is $x\phi(x)g(x) = o(1)$ $(C, \beta + 1)$, as $x \to \infty$.*

4 Tauber's second type theorems for distributional point values and (A) summability

4.1 Tauberian theorem for distributional point values.

We are ready to show the main theorem of this article.

THEOREM 4.1. *Let F be analytic in a rectangular region of the form $(a, b) \times (0, R)$. Suppose $f(x) = F(x+i0)$ in $\mathcal{D}'(a,b)$. Let $x_0 \in (a,b)$ such that $F(x_0+iy) \to \gamma$ as $y \to 0^+$. The Tauberian condition $f'(x_0 + \varepsilon x) = O(\varepsilon^{-1})$ as $\varepsilon \to 0^+$ in $\mathcal{D}'(a,b)$ implies that $f(x_0) = \gamma$, distributionally.*

PROOF. Clearly, by a translation, we can assume that $x_0 = 0$. We first show that it may be assumed $f \in \mathcal{S}'(\mathbb{R})$ and F is the standard Fourier-Laplace representation [2, 45]. Let C be a smooth simple curve contained in $(a, b) \times [0, R)$ such that $C \cap (a, b) = [x_0 - \sigma, x_0 + \sigma]$, for some small σ, and which is symmetric with respect to the imaginary axis. Let τ be a conformal bijection [28] between the upper half-plane and the region enclosed by C such that the image of the imaginary axis is contained on the imaginary axis and τ extends to a C^∞-diffeomorphism from $\mathbb{R}$ to $C \setminus (C \cap i\mathbb{R}_+)$. Then, $F \circ \tau(iy) \to \gamma$ as $y \to 0^+$, and $f(\tau(\varepsilon x)) = O(\varepsilon^{-1})$ as $\varepsilon \to 0^+$ in $\mathcal{D}'(\mathbb{R})$ if and only if $F(iy) \to \gamma$ and $f(\varepsilon x) = O(\varepsilon^{-1})$ in $\mathcal{D}'(\mathbb{R})$. Moreover, $f \circ \tau(0) = \gamma$ if and only if $f(0) = \gamma$, distributionally [21]. In addition $F \circ \tau$ is bounded away from an open half-disk about the origin, hence it is the Fourier-Laplace analytic representation of $f \circ \tau$. We can therefore assume that $f \in \mathcal{S}'(\mathbb{R})$ and

$$F(z) = \frac{1}{2\pi} \left\langle \hat{f}(t), e^{izt} \right\rangle .$$

Our aim is to show that f is distributionally bounded at $x = 0$. Indeed, if one established this fact then $f(0) = \gamma$, distributionally, by Theorem 2.1. The condition $f'(\varepsilon x) = O(\varepsilon^{-1})$ still holds in $\mathcal{S}'(\mathbb{R})$ [9, 38]. If we integrate this condition [43], we obtain from the definition of primitive in $\mathcal{S}'(\mathbb{R})$ that there exists a function c, continuous on $(0, \infty)$, such that

$$f(\varepsilon x) = c(\varepsilon) + O(1) ,$$

as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R})$, in the sense that for each $\phi \in \mathcal{S}(\mathbb{R})$,

$$\langle f(\varepsilon x), \phi(x) \rangle = c(\varepsilon) \int_{-\infty}^{\infty} \phi(x)\mathrm{d}x + O(1) ,$$

as $\varepsilon \to 0^+$. Fourier transforming the last relation, we have that

$$\hat{f}(\lambda x) = 2\pi c \left(\lambda^{-1} \right) \frac{\delta(x)}{\lambda} + O\left(\frac{1}{\lambda} \right) ,$$

as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$. Evaluating at e^{-x}, we obtain, as $y \to 0^+$, $F(iy) = O(1)$ and

$$F(iy) = \frac{1}{2\pi} \left\langle \hat{f}(t), e^{-yt} \right\rangle = \frac{1}{2\pi y} \left\langle \hat{f}\left(y^{-1}t \right), e^{-t} \right\rangle = c(y) + O(1) .$$

Hence, c is bounded near the origin, and thus $f(\varepsilon x) = c(\varepsilon) + O(1) = O(1)$ as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R})$, as required. $\qquad \square$

So, we obtain the following Tauberian theorem in terms of the Laplace transform.

THEOREM 4.2. *Let $G \in \mathcal{D}'(\mathbb{R})$ have the support bounded at the left. Necessary and sufficient conditions for*

$$\lim_{\lambda \to \infty} G(\lambda x) = \gamma \quad in \ \mathcal{D}'(\mathbb{R}) \ , \tag{14}$$

are

$$\lim_{y \to 0^+} y\mathcal{L}\{G; y\} = \lim_{y \to 0^+} \mathcal{L}\{G'; y\} = \gamma, \tag{15}$$

and

$$\lambda x G'(\lambda x) = O(1) \quad as \ \lambda \to \infty \ in \ \mathcal{D}'(\mathbb{R}). \tag{16}$$

PROOF. Either (14) or (16) implies that G is a tempered distribution and hence its Laplace transform is well defined for $\Re e\, z > 0$. The necessity is clear. Now, the condition (16) translates into $f'(\varepsilon x) = O(\varepsilon^{-1})$ in $\mathcal{S}'(\mathbb{R})$, where $\hat{f} = 2\pi G'$. Relation (25) gives $F(iy) = \mathcal{L}\{G'; y\} \to \gamma$ as $y \to 0^+$, for the Fourier-Laplace representation of f, hence by Theorem 4.1, $f(0) = \gamma$ in $\mathcal{S}'(\mathbb{R})$. Hence, taking Fourier inverse transform, we conclude that $G'(\lambda x) \sim \lambda^{-1}\gamma\delta(x)$ as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$, which implies (14) [40]. $\qquad\square$

4.2 Tauberian results for Abel limits.

Let us define Abel limits for distributions. Let $g \in \mathcal{D}'(\mathbb{R})$, it is called Laplace transformable on the strip $a < \Re e\, z < b$ if $e^{-\xi t}g(t)$ is a tempered distribution for $a < \xi < b$ (see [31]); in such a case its Laplace transform is well defined on that strip.

DEFINITION 4.3. Let $f \in \mathcal{D}'(\mathbb{R})$. We say that f has a limit γ at infinity in the Abel sense, and write

$$\lim_{x \to \infty} f(x) = \gamma \quad (A) \ ,$$

if there exists a distribution f_+ with support bounded at the left such that f_+ coincides with f on an open interval (a, ∞), f_+ is Laplace transformable for $\Re e\, z > 0$, and

$$\lim_{y \to 0^+} y\mathcal{L}\{f_+; y\} = \gamma \ . \tag{17}$$

Notice that Definition 4.3 is independent of the choice of f_+, because every compactly supported distribution satisfies (17) with $\gamma = 0$. The case of locally integrable functions is of interest; it is analyzed in the next example.

EXAMPLE 4.4. If $f \in L^1_{\text{loc}}[0, \infty)$ is such that the improper integral

$$\mathcal{L}\{f; y\} = \int_0^\infty f(t)e^{-ty}\mathrm{d}t \quad \text{converges for each } y > 0 \ , \tag{18}$$

and

(19)
$$\lim_{y \to 0^+} y\mathcal{L}\{f; y\} = \gamma ,$$

then f has γ as an Abel limit in the sense of Definition 4.3. However, the Abel limit of f, in the sense of Definition 4.3, exists under weaker assumptions, namely, under the existence of the Laplace transform as integrals in the Cesàro sense, i.e.,

(20)
$$\mathcal{L}\{f; y\} = \int_0^\infty f(t)e^{-ty}\mathrm{d}t \quad \text{(C)} \quad \text{exists for each } y > 0 ,$$

and (19). Observe that the order of (C) summability might change in (20) with each y. Conversely, the reader may verify that the existence of the Abel limit, interpreted as in Definition 4.3, of a locally integrable function is equivalent to (20) and (19).

Observe that (17) coincides with (15). Therefore, using the well-known equivalence between Cesàro behavior and parametric (quasiasymptotic) behavior [10, 37, 39], we may reformulate Theorem 4.2.

COROLLARY 4.5. *Let $f \in \mathcal{D}'(\mathbb{R})$. A necessary and sufficient condition for*

(21)
$$\lim_{x \to \infty} f(x) = \gamma \quad \text{(C)}$$

is

$$\lim_{x \to \infty} f(x) = \gamma \quad \text{(A)} \quad and \quad xf'(x) = O(1) \quad \text{(C)} , \quad as \; x \to \infty .$$

We now combine Corollary 4.5 with the results from Section 3.4 to obtain more precise information about the Cesàro order in (21).

THEOREM 4.6. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that $\lim_{x \to \infty} f(x) = \gamma$ (A). The Tauberian condition $xf'(x) = O(1)$ $(\text{C}, \beta_1 + 1)$, as $x \to \infty$, implies $\lim_{x \to \infty} f(x) = \gamma$ (C, β) for all $\beta > \beta_1$.*

PROOF. It follows directly from Corollary 4.5 and Theorem 3.10. □

We can also consider a one-sided Tauberian condition.

THEOREM 4.7. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that $\lim_{x \to \infty} f(x) = \gamma$ (A). Let $\beta_1 \geq 1$. The one-sided Tauberian condition $xf'(x) = O_L(1)$ (C, β_1), as $x \to \infty$, implies $\lim_{x \to \infty} f(x) = \gamma$ (C, β) for all $\beta \geq \beta_1$.*

PROOF. We may assume that f is real-valued and $\operatorname{supp} f \subseteq [1, \infty)$. If $xf'(x) = O(1)$ (C) is established, we could apply first Corollary 4.5 and then Theorem 3.11 to obtain the desired conclusion. By assumption, there exist $K > 0$ and $k \in \mathbb{N}$ such that $g^{(-k)} \in L_{\text{loc}}^\infty(\mathbb{R})$ and $g^{(-k)}(x) \geq -Kx^k$ for all $x \geq 1$, where $g(x) = xf'(x)$. We consider $T(x) = g^{(-k)}(x)/x$. If we show that $T(x) = O(x^{k-1})$ (C, 1), then it would immediately follow $g(x) = O(1)$ (C, $k + 1$), as

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required. By Lemma 3.8, we have the equality $T(x) = f^{(-k+1)}(x) - kf^{(-k)}(x)/x$; therefore, going to the Laplace transforms,

$$\mathcal{L}\{T;y\} = y^{-k}y\mathcal{L}\{f;y\} - k\int_y^\infty u^{-k}\mathcal{L}\{f;u\}\,\mathrm{d}u = \frac{\gamma}{y^k} - k\frac{\gamma}{ky^k} + o\left(\frac{1}{y^k}\right) = o\left(\frac{1}{y^k}\right) ,$$

$y \to 0^+$. Using now the inequality $0 \le Kx^{k-1} + T(x)$, we have

$$T^{(-1)}(x) = \int_1^x (T(t) + Kt^{k-1})\mathrm{d}t + O(x^k) \le e\int_0^x e^{-\frac{t}{x}}(T(t) + Kt^{k-1})\mathrm{d}t + O(x^k)$$

$$\le e\mathcal{L}\left\{T; \frac{1}{x}\right\} + O(x^k) = o(x^k) + O(x^k) = O(x^k) ;$$

hence $T(x) = O(x^{k-1})$ (C, 1). The proof is complete. $\qquad\square$

We obtain from Theorem 4.7 an extension of a classical important result of Szász [34, Thm.1].

THEOREM 4.8. *Let $f \in L_{\mathrm{loc}}^1[0,\infty)$. Suppose that $\lim_{x\to\infty} f(x) = \gamma$ (A) in the sense that it satisfies (20) and (19). Then, the one-sided Tauberian condition*

$$(22) \qquad\qquad xf(x) - \int_0^x f(t)\mathrm{d}t \ge -Kx, \quad x > a ,$$

for some positive constants K and a, implies that

$$(23) \qquad\qquad f^{(-1)}(x) = \int_0^x f(t)\mathrm{d}t \sim \gamma x, \quad x \to \infty .$$

PROOF. Note that the relations (22) and (23) exactly mean that $xf'(x) = O_L(1)$ (C, 1) and $\lim_{x\to\infty} f(x) = \gamma$ (C, 1), respectively. So, Theorem 4.7 yields (23). $\qquad\square$

In particular, we have the ensuing corollary.

COROLLARY 4.9 (Szász [34]). *Let $f \in L_{\mathrm{loc}}^1[0,\infty)$ satisfy (18) and (19). The one-sided Tauberian condition (22) implies (23).*

REMARK 4.10. If $\beta \ge 0$, we might replace (22) in Theorem 4.8 and Corollary 4.9 by

$$xf(x) - \int_0^x f(t)\mathrm{d}t = O_L(x) \quad (C, \beta), \quad \text{as } x \to \infty ,$$

then the same arguments apply to conclude $\lim_{x\to\infty} f(x) = \gamma$ (C, $\beta+1$).

We can use Theorem 3.12 to obtain a Tauber type characterization of (C, β) limits; the next result follows easily from Corollary 4.5 and Theorem 3.12.

THEOREM 4.11. *Let $f \in \mathcal{D}'(\mathbb{R})$ and $\beta \ge 0$. A necessary and sufficient condition for $\lim_{x\to\infty} f(x) = \gamma$ (C, β) is $\lim_{x\to\infty} f(x) = \gamma$ (A) and $xf'(x) = o(1)$ (C, $\beta+1$).*

4.3 Tauberians for Abel summability of distributional evaluations.

In order to give applications to the classical cases, let us give sense to Abel summability of distributional evaluations [10]. Let $g \in \mathcal{D}'(\mathbb{R})$ with the support bounded at the left and $\phi \in \mathcal{E}(\mathbb{R})$. We say that the distributional evaluation $\langle g(x), \phi(x) \rangle$ exists and equals γ in the Abel sense if $e^{-yx}\phi(x)g(x) \in \mathcal{S}'(\mathbb{R})$ for every $y > 0$ and

$$\lim_{y \to 0^+} \left\langle \phi(t)g(t), e^{-yt} \right\rangle = \gamma \ .$$

In such a case we write

$$(24) \qquad\qquad\qquad \langle g(x), \phi(x) \rangle = \gamma \quad (\mathrm{A}) \ .$$

Notice that (24) holds if and only if $\lim_{x \to \infty} G(x) = \gamma$ (A), where G is the first order primitive of ϕg with the support bounded at the left, that is, $G = (\phi g)^{(-1)} = (\phi g) * H$ (here H is the Heaviside function). So, our theorems from Section 4.2 give at once the following results.

THEOREM 4.12. *Let $g \in \mathcal{D}'(\mathbb{R})$ with the support bounded at the left and $\phi \in \mathcal{E}(\mathbb{R})$. Suppose that*

$$(25) \qquad\qquad\qquad \langle g(x), \phi(x) \rangle = \gamma \quad (\mathrm{A}) \ .$$

The Tauberian condition $xg(x)\phi(x) = O_L(1)$ $(\mathrm{C}, \beta_1 + 1)$, as $x \to \infty$, for $\beta_1 \geq 0$, implies

$$(26) \qquad\qquad\qquad \langle g(x), \phi(x) \rangle = \gamma \quad (\mathrm{C}, \beta)$$

for all $\beta \geq \beta_1 + 1$. Moreover, the stronger Tauberian condition $xg(x)\phi(x) = O(1)$ $(\mathrm{C}, \beta_1 + 1)$ implies that (26) holds for all $\beta > \beta_1$.

THEOREM 4.13. *Let $g \in \mathcal{D}'(\mathbb{R})$ with the support bounded at the left and $\phi \in \mathcal{E}(\mathbb{R})$. A necessary and sufficient condition for (26) is $\langle g(x), \phi(x) \rangle = \gamma$ (A) and $xg(x)\phi(x) = o(1)$ $(\mathrm{C}, \beta + 1)$ as $x \to \infty$.*

The case when $g = \hat{f}$ and $\phi(x) = e^{ix_0 x}$ is interesting, since it provides the order of summability in the pointwise Fourier inversion formula for Łojasewicz point values [40]. This is the content of the next corollary.

COROLLARY 4.14. *Let $f \in \mathcal{S}'(\mathbb{R})$ be such that $\operatorname{supp} \hat{f}$ is bounded at the left and*

$$\frac{1}{2\pi} \left\langle \hat{f}(x), e^{ix_0 x} \right\rangle = \gamma \quad (\mathrm{A}) \ .$$

Then $xe^{ix_0 x}\hat{f}(x) = O_L(1)$ $(\mathrm{C}, \beta_1 + 1)$, for some $\beta_1 \geq 0$, implies that $f(x_0) = \gamma$, distributionally. Moreover, the pointwise Fourier inversion formula holds in the (C, β) sense for any $\beta \geq \beta_1 + 1$, that is,

$$(27) \qquad\qquad f(x_0) = \gamma, \ distributionally, \ and \ \frac{1}{2\pi} \left\langle \hat{f}(x), e^{ix_0 x} \right\rangle = \gamma \quad (\mathrm{C}, \beta) \ .$$

Furthermore, the stronger Tauberian condition $xe^{ix_0 x}\hat{f}(x) = O(1)$ $(\mathrm{C}, \beta_1 + 1)$ implies that (27) holds for all $\beta > \beta_1$.

4.4   Tauberians for series and Stieltjes integrals.

The cases of Stieltjes integrals and series are also of importance. We obtain directly from Theorem 4.12 the following corollary.

COROLLARY 4.15. *Let s be a function of local bounded variation such that $s(x) = 0$ for $x \leq 0$. Suppose that the Cesàro integral*

$$(28) \qquad \mathcal{L}\{ds; y\} = \int_0^\infty e^{-yx} ds(x) \quad (C) \quad \text{exists for each } y > 0 ,$$

and that

$$\lim_{y \to 0^+} \mathcal{L}\{ds; y\} = \gamma .$$

Let $\beta_1 \geq 0$. Then, the Tauberian condition

$$(29) \qquad \int_0^x t\, ds(t) = O_L(x) \quad (C, \beta_1)$$

implies that for all $\beta \geq \beta_1 + 1$,

$$(30) \qquad \lim_{x \to \infty} s(x) = \gamma \quad (C, \beta) .$$

Moreover, if we replace $O_L(x)$ by $O(x)$ in (29), we conclude that (30) holds for all $\beta > \beta_1$.

Observe that in particular Corollary 4.15 holds if we replace (28) by the stronger assumption of the existence of the improper integrals $\int_0^\infty e^{-yx} ds(x) = \lim_{t \to \infty} \int_0^t e^{-yx} ds(x)$ for each $y > 0$.

Let $\lambda_n \nearrow \infty$ be an increasing sequence of non-negative real numbers. Recall that we write $\sum_{n=0}^\infty c_n = \gamma \ (A, \{\lambda_n\})$ if the Dirichlet series $F(z) = \sum_{n=0}^\infty c_n e^{-z\lambda_n}$ converges on $\Re e\, z > 0$ and $\lim_{y \to 0^+} F(y) = \gamma$ (see [12]).

COROLLARY 4.16. *Suppose that $\sum_{n=0}^\infty c_n = \gamma \ (A, \{\lambda_n\})$. The condition $\sum_{\lambda_n < x} \lambda_n c_n = O_L(x)$ (C, β_1), for $\beta_1 \geq 0$, implies that $\sum_{n=0}^\infty c_n = \gamma \ (R, \{\lambda_n\}, \beta)$ for all $\beta \geq \beta_1 + 1$. The stronger Tauberian condition $\sum_{\lambda_n < x} \lambda_n c_n = O(x) \ (C, \beta_1)$ implies the $(R, \{\lambda_n\}, \beta)$ summability of the series to γ for all $\beta > \beta_1$.*

Furthermore, we may formulate a much stronger version of Corollary 4.16 if we assume

$$F(y) = \sum_{n=0}^\infty c_n e^{-y\lambda_n} \quad (R, \{\lambda_n\}) \quad \text{exists for each } y > 0 ,$$

and $\lim_{y \to 0^+} F(y) = \gamma$, instead of the more restrictive hypothesis of $(A, \{\lambda_n\})$ summability. On the other hand, if we specialize Corollary 4.16 to power series, we now obtain a general form of the Theorem 1.3 stated at the introduction.

COROLLARY 4.17. *Suppose that $\sum_{n=0}^\infty c_n = \gamma \ (A)$. The Tauberian condition $\sum_{n=1}^N n c_n = O_L(N) \ (C, \beta_1)$, for $\beta_1 \geq 0$, implies that $\sum_{n=0}^\infty c_n = \gamma \ (C, \beta)$ for all $\beta \geq \beta_1 + 1$. The stronger Tauberian condition $\sum_{n=1}^N n c_n = O(N) \ (C, \beta_1)$ implies the (C, β) summability of the series to γ for all $\beta > \beta_1$.*

5 Applications: Tauberian conditions for convergence

This section is devoted to applications of the distributional method in classical Tauberians for Dirichlet series. Let $f \in \mathcal{D}'(\mathbb{R})$ have support bounded at the left. As follows from the results of [10, 40], we have $\lim_{x\to\infty} f(x) = \gamma$ (C) if and only if its derivative has the following quasiasymptotic behavior

$$(31) \qquad f'(\lambda x) = \gamma \frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) .$$

Let $1 < \sigma < 2$. Throughout this section $\phi_\sigma \in \mathcal{D}(\mathbb{R})$ is a fixed test function with the following properties: $0 \leq \phi_\sigma \leq 1$, $\phi_\sigma(x) = 1$ for $x \in [0,1]$, and $\operatorname{supp} \phi_\sigma \subseteq [-1, \sigma]$.

We first extend a Theorem of Szász [35] (see also [30]) from series to Stieltjes integrals.

THEOREM 5.1. *Let s be a function of local bounded variation such that $s(x) = 0$ for $x \leq 0$. Suppose that $\lim_{x\to\infty} s(x) = \gamma$ (A). Then, the Tauberian conditions*

$$(32) \qquad \int_0^x t \, \mathrm{d}s(t) = O_L(x) \quad (C, \beta) ,$$

for some $\beta \geq 0$, and

$$(33) \qquad \lim_{\sigma \to 1^+} \limsup_{x\to\infty} \frac{1}{x} \int_x^{\sigma x} t \, |\mathrm{d}s|(t) = 0$$

imply that $\lim_{x\to\infty} s(x) = \gamma$.

PROOF. Corollary 4.15 and (32) imply that $\lim_{x\to\infty} s(x) = \gamma$ (C). Then s' has the quasiasymptotic behavior (31), evaluating the quasisymptotics at ϕ_σ, we obtain

$$\limsup_{\lambda\to\infty} |s(\lambda) - \gamma| \leq \limsup_{\lambda\to\infty} \int_\lambda^{\sigma\lambda} \phi_\sigma\left(\frac{t}{\lambda}\right) |\mathrm{d}s|(t)$$

$$\leq \limsup_{\lambda\to\infty} \frac{1}{\lambda} \int_\lambda^{\sigma\lambda} t \, |\mathrm{d}s|(t) .$$

Since σ is arbitrary, we obtain the convergence from (33). $\qquad\square$

We recover the result of Szász mentioned above.

COROLLARY 5.2 (Szász, [35]). *Suppose that $\sum_{n=0}^\infty c_n = \gamma$ (A). Then the Tauberian conditions*

$$(34) \qquad V_N = \frac{1}{N} \sum_{n=0}^N n \, |c_n| = O(1) ,$$

and

$$(35) \qquad V_m - V_n \to 0, \quad as \; \frac{m}{n} \to 1^+ \text{ and } n \to \infty ,$$

imply the convergence of the series to γ.

PROOF. We show that (35) implies (33). Indeed, by (34),

$$\frac{1}{x} \sum_{x < n \leq \sigma x} n\, |c_n| = \frac{[\sigma x] - [x]}{x} V_{[\sigma x]} + \frac{[x]}{x}(V_{[\sigma x]} - V_{[x]})$$

$$< \frac{\sigma x - x - 1}{x} O(1) + (V_{[\sigma x]} - V_{[x]}) \, ,$$

and the last expression tends to 0 as $x \to \infty$ and $\sigma \to 1^+$. $\qquad\square$

The next Tauberian theorem for Dirichlet series belongs to Hardy and Littlewood [14] (see also [33] and [34, Thm. 6]).

THEOREM 5.3 (Hardy-Littlewood). *Suppose that*

$$\sum_{n=0}^{\infty} c_n = \gamma \quad (\mathrm{A}, \{\lambda_n\}) \, .$$

Then the Tauberian condition

(36)
$$\sum_{n=1}^{\infty} \left(\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} \right)^{p-1} |c_n|^p < \infty \, ,$$

where $1 \leq p < \infty$, implies the convergence of the series to γ.

PROOF. Since the case $p = 1$ is trivial, we assume $1 < p < \infty$. Let $q = p/(p-1)$. Hölder's inequality implies (32), with $\beta = 0$, for $s(x) = \sum_{\lambda_n \leq x} c_n$. So, Corollary 4.16 implies the $(\mathrm{R}, \{\lambda_n\}), 1)$ summability. Then $\sum_{n=0}^{\infty} c_n \delta(x - \lambda_n)$ has the quasiasymptotic behavior (31), evaluating at ϕ_σ and using Hölder's inequality, we obtain

$$\limsup_{N \to \infty} \left| \sum_{n=0}^{N} c_n - \gamma \right| \leq \limsup_{N \to \infty} \sum_{\lambda_N < \lambda_n \leq \sigma \lambda_N} \phi_\sigma \left(\frac{\lambda_n}{\lambda_N} \right) |c_n|$$

$$\leq \limsup_{N \to \infty} \sum_{\lambda_N < \lambda_n \leq \sigma \lambda_N} |c_n|$$

$$\leq \limsup_{N \to \infty} \left(\sum_{\lambda_N < \lambda_n \leq \sigma \lambda_N} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \right)^{1/q} O(1)$$

$$\leq (\sigma - 1)^{1/q} O(1) \, ,$$

and by taking $\sigma \to 1^+$, we obtain the result. $\qquad\square$

We end this article by proving another theorem of Szász [32, 33, 34] (the case for power series was discovered first by Hardy and Littlewood).

THEOREM 5.4 (Szász, [34]). *Suppose that $\sum_{n=0}^{\infty} c_n = \gamma$ $(\mathrm{A}, \{\lambda_n\})$. Then the Tauberian condition*

(37)
$$\sum_{n=1}^{N} \lambda_n^p (\lambda_n - \lambda_{n-1})^{1-p} |c_n|^p = O(\lambda_N) \, ,$$

for some $1 < p < \infty$, implies the convergence of the series to γ.

PROOF. Let $q = p/(p-1)$. Again, Hölder's inequality implies (32), with $\beta = 0$, for $s(x) = \sum_{\lambda_n \leq x} c_n$. So, Corollary 4.16 implies that $\sum_{n=0}^{\infty} c_n \delta(x - \lambda_n)$ has the quasiasymptotic behavior (31), and by evaluating at ϕ_σ and using Hölder's inequality, we obtain

$$
\begin{aligned}
\limsup_{N \to \infty} \left| \sum_{n=0}^{N} c_n - \gamma \right| &\leq \limsup_{N \to \infty} \sum_{\lambda_N < \lambda_n \leq \sigma \lambda_N} \phi_\sigma \left(\frac{\lambda_n}{\lambda_N} \right) |c_n| \\
&\leq \limsup_{N \to \infty} \lambda_N^{1/p} \left(\sum_{\lambda_N < \lambda_n \leq \sigma \lambda_N} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n^q} \right)^{1/q} O(1) \\
&\leq \limsup_{N \to \infty} \left(\frac{1}{\lambda_N} \sum_{\lambda_N < \lambda_n \leq \sigma \lambda_N} (\lambda_n - \lambda_{n-1}) \right)^{1/q} O(1) \\
&= (\sigma - 1)^{1/q} O(1) .
\end{aligned}
$$

Since σ is arbitrary, we obtain the convergence. $\qquad\square$

REMARK 5.5. We have chosen to give direct proofs of Theorems 5.3 and 5.4 to exemplify the distributional method. However, they are easy consequences of Theorem 5.1. Namely, by applying Hölder's inequality to $\sum_{x < \lambda_n \leq \sigma x} \lambda_n |c_n|$, one deduces that (33) is satisfied. Indeed, any of the Tauberian hypotheses (36) or (37) yield

$$
\frac{1}{x} \sum_{x < \lambda_n \leq \sigma x} \lambda_n |c_n| \leq \sigma^{1/p} (\sigma - 1)^{1/q} O(1) .
$$

References

[1] K. ANANDA RAU, On the converse of Abel's theorem, J. London Math. Soc. 3 (1928), 349–355.

[2] H. BREMERMANN, Distributions, complex variables and Fourier transforms, Addison-Wesley, Reading, Massachusetts, 1965.

[3] K. CHANDRASEKHARAN AND S. MINAKSHISUNDARAM, Typical means, Oxford University Press, Oxford, 1952.

[4] F. CONSTANTINESCU, Boundary values of analytic functions, Comm. Math. Phys. 7 (1968), 225–233.

[5] JU. N. DROŽŽINOV AND B. I. ZAV'JALOV, Quasiasymptotic behavior of generalized functions, and Tauberian theorems in the complex domain, (Russian) Mat. Sb. (N.S.) 102 (144) (1977), 372–390.

[6] YU. N. DROZHZHINOV AND B. I. ZAV'YALOV, Theorems of Hardy-Littlewood type for measures of variable sign in a cone, Sb. Math. 186 (1995), 675–693.

[7] R. Estrada, The Cesàro behaviour of distributions, Proc. Roy. Soc. London Ser. A 454 (1998), 2425–2443.

[8] R. Estrada, Boundary values of analytic functions without distributional point values, Tamkang J. Math. 35 (2004), 53–60.

[9] R. Estrada, A distributional version of the Ferenc Lukács theorem, Sarajevo J. Math. 1 (2005), 1–17.

[10] R. Estrada and R. P. Kanwal, A distributional approach to asymptotics. Theory and applications, Second edition, Birkhäuser, Boston, 2002.

[11] R. Estrada and J. Vindas, On the point behavior of Fourier series and conjugate series, Z. Anal. Anwend. 29 (2010), 487–504.

[12] G. H. Hardy, Divergent Series, Clarendon Press, Oxford, 1949.

[13] G. H. Hardy and J. E. Littlewood, Contributions to the arithmetic theory of series, Proc. London Math. Soc. 11 (1913), 411–478.

[14] G. H. Hardy and J. E. Littlewood, Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive, Proc. London Math. Soc. 13 (1914), 174–191.

[15] G. H. Hardy and J. E. Littlewood, Some theorems concerning Dirichlet's series, Messenger of Mathematics 43 (1914), 134–147.

[16] A. E. Ingham, The equivalence theorem for Cesàro and Riesz summability, Publ. Ramanujan Inst. 1 (1968), 107–113.

[17] J. Karamata, Über die Hardy-Littlewoodschen umkehrungen des Abelschen stetigkeitssatzes, Math. Z. 32 (1930), 319–320.

[18] J. Karamata, Neuer beweis und verallgemeinerung der Tauberschen sätze, welche die Laplacesche und Stieltjessche transformation betreffen, J. Reine Angew. Math. 164 (1931), 27–39.

[19] J. Korevaar, Tauberian theory. A century of developments, Grundlehren der Mathematischen Wissenschaften, 329, Springer-Verlag, Berlin, 2004.

[20] J. E. Littlewood, The converse of Abel's theorem on power series, Proc. London Math. Soc. 9 (1911), 434–448.

[21] S. Łojasiewicz, Sur la valuer et la limite d'une distribution en un point, Studia Math. 16 (1957), 1–36.

[22] O. P. Misra and J. Lavoine, Transform analysis of generalized functions, North-Holland, Amsterdam, 1986.

[23] J. Peetre, On the value of a distribution at a point, Portugal Math. 27 (1968), 149–159.

[24] S. Pilipović and B. Stanković, Wiener Tauberian theorems for distributions, J. London Math. Soc. 47 (1993), 507–515.

[25] S. Pilipović and B. Stanković, Tauberian theorems for integral transforms of distributions, Acta Math. Hungar. 74 (1997), 135–153.

[26] S. Pilipović, B. Stanković and A. Takači, Asymptotic behaviour and Stieltjes transformation of distributions, Teubner-Texte zur Mathematik, Leipzig, 1990.

[27] S. Pilipović, B. Stanković and J. Vindas, Asymptotic behavior of generalized functions, Series on Analysis, Applications and Computations, 5, World Scientific Publishing Co., Hackensack, New Jersey, 2011.

[28] Ch. Pommerenke, Boundary behaviour of conformal maps, Springer-Verlag, Berlin, 1992.

[29] C. T. Rajagopal, Note on some Tauberian theorems of O. Szász, Pacific J. Math. 2 (1952), 377–384.

[30] A. Rényi, On a Tauberian theorem of O. Szász, Acta Univ. Szeged. Sect. Sci. Math. 11 (1946), 119–123.

[31] L. Schwartz, Théorie des distributions, Hermann, Paris, 1966.

[32] O. Szász, Verallgemeinerung eines Littlewoodschen satzes über potenzreihen, J. London Math. Soc. 3 (1928), 254–262.

[33] O. Szász, Generalization of two theorems of Hardy and Littlewood on power series, Duke Math. J. 1 (1935), 105–111.

[34] O. Szász, Converse theorems of summability for Dirichlet's series, Trans. Amer. Math. Soc. 39 (1936), 117–130.

[35] O. Szász, On a Tauberian theorem for Abel summability, Pacific J. Math. 1 (1951), 117–125.

[36] A. Tauber, Ein satz aus der theorie der unendlichen reihen, Monatsh. Math. Phys. 8 (1897), 273–277.

[37] J. Vindas, Structural theorems for quasiasymptotics of distributions at infinity, Publ. Inst. Math. (Beograd) (N.S.) 84 (98), 159–174.

[38] J. Vindas, Local behavior of distributions and applications, Dissertation, Louisiana State University, Baton Rouge, 2009.

[39] J. VINDAS, The structure of quasiasymptotics of Schwartz distributions, Linear and non-linear theory of generalized functions and its applications, pp. 297–314, Banach Center Publ. 88, Polish Acad. Sci. Inst. Math., Warsaw, 2010.

[40] J. VINDAS AND R. ESTRADA, Distributional point values and convergence of Fourier series and integrals, J. Fourier. Anal. Appl. 13 (2007), 551–576.

[41] J. VINDAS AND R. ESTRADA, A Tauberian theorem for distributional point values, Arch. Math. (Basel) 91 (2008), 247–253.

[42] J. VINDAS AND R. ESTRADA, On the order of summability of the Fourier inversion formula, Anal. Theory Appl. 26 (2010), 13–42.

[43] J. VINDAS AND S. PILIPOVIĆ, Structural theorems for quasiasymptotics of distributions at the origin, Math. Nachr. 282 (2009), 1584–1599.

[44] V. S. VLADIMIROV, A multidimensional generalization of the Tauberian theorem of Hardy and Littlewood, Math. USSR-Izv. 10 (1976), 1031–1048.

[45] V. S. VLADIMIROV, Methods of the theory of generalized functions, Taylor & Francis, London, 2002.

[46] V. S. VLADIMIROV, Yu. N. DROZHZHINOV AND B. I. ZAVIALOV, Tauberian theorems for generalized functions, Kluwer Academic Publishers, Dordrecht, 1988.

[47] D. V. WIDDER, The Laplace transform, Princeton Univ. Press, Princeton, 1941.

[48] N. WIENER, Tauberian theorems, Ann. of Math. 33 (1932), 1–100.

DEPARTMENT OF MATHEMATICS
LOUISIANA STATE UNIVERSITY
BATON ROUGE
LA 70803
USA

E-mail address: restrada@math.lsu.edu

DEPARTMENT OF MATHEMATICS
GHENT UNIVERSITY
KRIJGSLAAN 281 GEBOUW S22
B 9000 GENT
BELGIUM

E-mail address: jvindas@cage.Ugent.be