A generalization of the Wiener-Ikehara theorem

(by Jasson Vindas, February 21, 2023, Analysis seminar, Ghent)

1 Introduction:

The Wiener-Ikehara theorem is a cornerstone in complex Tauberian theory. Recently, Toshihiro Koga has given an interesting generalization of this theorem that only makes use of the real part of the Laplace transform in its hypothesis.

Let us first discuss the classical W-I theorem, which in its simplest form states:

Theorem 1 (W-I, 1933): Let $S \nearrow$ on $[0,\infty)$ and have convergent Laplace transform $\mathcal{L}\{S;s\} = \int_0^\infty S(x)e^{-sx}dx$ for $\operatorname{Re} s > 1$. If there is $A \geq 0$ such that

$$\mathcal{L}\{S;s\} - \frac{A}{s-1} \text{ has analytic continuation}$$

across $\operatorname{Re} s = 1$, then

$$(1) \qquad S(x) \sim Ae^x$$

A change of variables and integration by parts leads to:

Theorem 2: Let $\lambda_n \nearrow \infty$ and $a_n \geq 0$. Suppose that $D(s) = \sum_{n=0}^{\infty} \frac{a_n}{\lambda_n^s}$ converges for $\operatorname{Re} s = 1$ and $D(s) - \frac{A}{s-1}$ can be analytically extended beyond $\operatorname{Re} s = 1$, then

$$\sum_{n \leq x} a_n \sim Ax, \quad x \to \infty.$$

The following is a typical application (and actually the original motivation to show the theorem).

<u>Application 1</u> (Deduction of PNT from the W-I theorem)

The PNT (prime number theorem) states that

$$\pi(x) = \sum_{p\leq x} 1 \sim \frac{x}{\log x} \iff \psi(x) := \sum_{n\leq x} \Lambda(n) \sim x,$$

where $\Lambda(n) = \begin{cases} \log p, & n = p^{\alpha}, \ p \text{ prime } \& \ \alpha > 0 \\ 0, & \text{otherwise} \end{cases}$.

We work with $\psi(x) \sim x$. It is known that $\displaystyle\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}$

where $\zeta(s) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta function.

Now, it is well-known (this is the extra ingredient used here) that $(s-1)\zeta(s)$ has analytic continuation beyond $\operatorname{Re} s = 1$ (actually it is entire) & does not vanish on $\operatorname{Re} s = 1$, therefore,

$$\frac{d}{ds}\Big(\log (s-1)\zeta(s)\Big) = \frac{1}{s-1} + \frac{\zeta'(s)}{\zeta(s)}, \text{ and consequently,}$$

the W-I theorem applies to deliver $\displaystyle\sum_{n\leq x} \Lambda(n) \sim x$. ////

The most general version of the W-I theorem delivering (1) is due to G. Debruyne and myself and makes use of

<u>Definition 1</u> S is log-linearly decreasing if

$$\liminf_{\delta\to 0^+} \liminf_{x\to\infty} \inf_{0\leq h\leq \delta} \frac{S(x+h)-S(x)}{e^x} \geq 0,$$

that is $(\forall \varepsilon > 0)(\exists \delta, x_0 > 0)$ s.t.

$$S(x+h) - S(x) \geq -\varepsilon e^x, \quad x \geq x_0. ///$$

Definition 2

① A distribution $g \in PF_{loc}(\mathbb{R})$ (local pseudo-function) if on each finite open interval I it coincides with a tempered distribution f_I such that $\hat{f}_I \in C_0(\mathbb{R})$ (continuous function vanishing at $\pm\infty$).

② A harmonic function $U(s)$ on $\operatorname{Re} s > \alpha$ is said to have PF_{loc}-behavior on $\operatorname{Re} s = \alpha$ if

$$g = \lim_{\sigma \to \alpha^+} U(\sigma + it), \text{ distributionally},$$

for some $g \in PF_{loc}(\mathbb{R})$. ///

Theorem 3: (Debruyne–V., 2016) $S(x) \sim Ae^x$

① S is log-linearly decreasing.

② $\mathcal{L}\{S;s\}$ converges for $\operatorname{Re} s > 1$ and $\mathcal{L}\{S;s\} - \dfrac{A}{s-1}$ has PF_{loc}-behavior on $\operatorname{Re} s = 1$. ///

The interesting generalization of Koga is the following one:

Theorem 4 (Koga, 2021) Suppose that S is log-linearly decreasing and that

$$(2) \qquad \int_0^\infty S(x) \frac{e^{-x}}{1+x^2}\, dx < +\infty.$$

Set $U(s) := \operatorname{Re} \mathcal{L}\{S;s\}$ for $\operatorname{Re} s > 1$. If

(i) U has L^1_{loc}-boundary values in the open boundary set $1 + i(\mathbb{R} \setminus \{0\})$, and

(ii) there are $\eta, \sigma_0 > 0$ and $g \in L^1(-\eta, \eta)$ such that

$$U(\sigma+it) \geq g(t) \text{ for } \sigma+it \in]1,\sigma_0[\times]-\eta,\eta[$$

then $S(x) \sim A e^x$, for some A.

<u>Remark 1</u> One can show that $A = \lim_{\sigma \to 1^+} (\sigma-1)U(\sigma)$.

The motivation of Koga was to give a new proof (and a generalization!) of the following theorem of Erdős, Feller, and Pollard (1949), the so-called basic Limit Theorem of renewal theory.

<u>Application 2</u> Let $q_n \in [0,1]$ such that $q_0 = 0$, $\sum_{n=1}^{\infty} q_n = 1$ and $\gcd\{n : q_n \neq 0\} = 1$. If p_n is defined as $p_0 = 1$, $p_n = \sum_{k=0}^{n-1} p_k q_{n-k}$, then

$$\lim_{n \to +\infty} p_n = \frac{1}{\sum_{k=1}^{\infty} k q_k},$$

for a deduction from Theorem 4, see Koga's article.

Our generalization of Koga's result is the following one, we will see in the next section how Theorem 4 follows from it.

<u>Theorem 5</u> Let f be linearly slowly decreasing and suppose its Laplace transform converges for $\operatorname{Re} s > 1$. If $U(s) = \operatorname{Re} \mathcal{L}\{S; s\}$ satisfies

① For any finite interval $\sup_{1<\sigma\leq 2} \int_a^b |U(\sigma+it)| \, dt$
$<\infty$.

② on each $0\notin(a,b)$, $\lim_{\sigma\to 1} U(\sigma+it) \in L^1(a,b)$.

then $S(x) \sim Ae^x$ for some $A>0$. ///

Instead of giving a proof of Theorem 5, we will show a version of it for power series that exemplifies the main idea we use.

$\boxed{2}$ Deducing Theorem 4 from Theorem 5.

The only thing we have to show is that

$$\int_0^\infty S(x) \frac{e^{-x}}{1+x^2} dx < \infty \text{ and } U(\sigma+it) \geq g(t)$$

lead to hypothesis ① in Theorem 5. Clearly we just have to show the condition on intervals $(-\eta,\eta)$, $\eta>0$.

For it we take $\phi\geq 0$ being 1 on $(-\eta,\eta)$ and having compactly support and $\hat{\phi}\geq 0$. Then, by Parseval's,

$$\int_{-\infty}^\infty \mathcal{L}\{S,\sigma+it\} \phi(t)\,dt = \int_0^\infty S(x) e^{-\sigma x} \hat{\phi}(x)\,dx$$

So that,

$$\int_{-\infty}^\infty U(\sigma+it)\phi(t)\,dt = \int_0^\infty S(x) e^{-\sigma x} \hat{\phi}(x)\,dx$$
$$= O(1), \quad 1\leq\sigma\leq 2$$

since $\int_{-\infty}^{\infty} |S(x)| e^{-\sigma x} \frac{dx}{1+x^2} \to \int_{-\infty}^{\infty} \frac{|S(x)| e^{-x} dx}{1+x^2}$,

by Beppo-Levi theorem.

Therefore,

$$\int_{-\eta}^{\eta} |U(r+it)| dt \leq \int_{-\infty}^{\infty} |g(t)| \phi(t) dt + \int_{-\infty}^{\infty} (U(r+it) - g(t)) \phi(t) dt \leq 2\int_{-\infty}^{\infty} |g(t)| dt + \int_{-\infty}^{\infty} U(r+it) \phi(t) dt = O(1).$$

$\boxed{3}$ A version of Theorem 5 for power series

Theorem 6: Suppose that $F(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < 1$, a_n real sequence. Set $U = \operatorname{Re} F$. If

(1) $\sup_{0<r<1} \int_{-\pi}^{\pi} |U(re^{i\theta})| d\theta < \infty$ and there is g s.t.

(2) $\lim_{r \to 1^-} U(re^{i\theta}) = g(\theta)$ in $L^1_{loc}([-\pi, \pi] \setminus \{0\})$,

then $\lim_{r \to 1^-} a_n$ exists.

Proof. Condition 1 leads to $U(re^{i\theta}) \to d\mu(\theta)$ weak* for some measure μ, (2) says that μ is absolutely continuous with respect to the Lebesgue measure off 0, therefore $d\mu = g + A\delta$, where δ is the Dirac

$d\theta$. Now,

$$a_n r^n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(re^{i\theta})\, e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi} 2 \int_{-\pi}^{\pi} F(re^{i\theta}) \cos n\theta\, d\theta$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} U(re^{i\theta}) \cos n\theta\, d\theta.$$

Taking $r \to 1^-$, we get

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos n\theta\, d\mu(\theta) = \frac{A}{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} \cos n\theta\, g(\theta)\, d\theta$$

$$= \frac{A}{\pi} + o(1),\ n \to \infty$$

by the Riemann-Lebesgue lemma.