A General Integral

Jasson Vindas
jvindas@cage.ugent.be

Department of Mathematics
Ghent University

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In this lecture we present the construction of a new integral for functions of one variable $f : [a, b] \to \mathbb{R}$.

We also present a brief overview of some standard integrals.

The integration theory to be presented is a collaborative work with R. Estrada.
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Introduction

The main drawbacks of the Riemann integral are:

1. The class of Riemann integrable functions is too small.
2. Lack of convergence theorems.
3. The fundamental theorem of calculus

\[ \int_a^x f(t)\,dt = F(x) \]

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- It is more general than the Lebesgue integral.
- The fundamental theorem of calculus is always valid.

For example, Denjoy integral integrates

\[ \int_0^1 \frac{1}{x} \sin \left( \frac{1}{x^2} \right) \, dx \]

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The integral that we shall construct has the following properties:

1. It is **more general** than the Denjoy-Perron-Henstock integral, and in particular than the Lebesgue integral.

2. It identifies a **new class of functions** with Schwartz distributions.

3. It enjoys all useful properties of the standard integrals, including:
   - Convergence theorems.
   - Integration by parts and substitution formulas.
   - Mean value theorems.

4. If $\beta > 0$, it integrates unbounded functions such as

   $\frac{1}{|x|^\gamma} \sin \left( \frac{1}{|x|^{\beta}} \right)$

   for all $\gamma \in \mathbb{R}$

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Outline

1. The integrals of Denjoy, Perron, and Henstock
   - Denjoy integral
   - Perron integral
   - Henstock-Kurzweil integral

2. From Denjoy to Łojasiewicz
   - Integration of higher order differential coefficients
   - Łojasiewicz point values

3. The Distributional Integral
   - Construction
   - Properties
   - Examples
In the construction of his integral, Denjoy developed a complicated procedure that he called “totalization”. He made use of transfinite induction. It is very well explained in Hobson’s book:


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In 1932, Romanovski eliminated the use of transfinite numbers from Denjoy’s construction.
In 1914, Perron developed another approach which is equivalent to the Denjoy integral.

**Definition**

Let $f : [a, b] \to \overline{\mathbb{R}}$.

1. $U$ is a (continuous) major function of $f$ if it is continuous on $[a, b]$, $U(a) = 0$, and

   $$f(x) \leq DU(x) \quad \text{and} \quad -\infty < DU(x), \ \forall x \in [a, b].$$

2. $V$ is a (continuous) minor function of $f$ if it is continuous on $[a, b]$, $V(a) = 0$, and

   $$DV(x) \leq f(x) \quad \text{and} \quad DV(x) < \infty, \ \forall x \in [a, b].$$
Perron integral

Definition

A function \( f : [a, b] \rightarrow \overline{\mathbb{R}} \) is said to be **Perron integrable** on \([a, b]\) if it has at least one major and one minor function and the numbers

\[
\inf \{ U(b) : U \text{ is continuous major function of } f \}
\]

\[
\sup \{ V(b) : V \text{ is continuous minor function of } f \}
\]

are equal and finite. The **common value** is said to be its **Perron integral**.
Henstock-Kurzweil integral

In the 1950’s Kurzweil introduced an integral which was motivated by his study in differential equations. His integral coincides with the Denjoy-Perron integral and it was systematically studied by Henstock during the 1960’s.

Interestingly, the definition of Henstock-Kurzweil integral does not differ much from that of Riemann integral. It is explained in detail in the monographs by Bartle and Gordon:


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A function $\delta : [a, b] \to \mathbb{R}_+$ is said to be a gauge on $[a, b]$. If $P = \{l_j\}_{j=1}^n$ is a partition of $[a, b]$ such that for each $l_j$ there is assigned a point $t_j \in l_j$, then we call $t_j$ a tag of $l_j$. We say that the partition is tagged and write

$$\dot{P} = \{(l_j, t_j)\}_{j=1}^n.$$

$\dot{P}$ is said to be $\delta$-fine if $l_j \subset [t_j - \delta(t_j), t_j + \delta(t_j)]$. 

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Henstock integral

Definition

Given a tagged partition \( \dot{P} := \{(l_j, t_j)\}_{j=1}^n \), we denote the Riemann sum of \( f \) corresponding to \( \dot{P} \) as

\[
S(f; \dot{P}) = \sum_{j=1}^{n} f(t_j)\ell(l_j) \quad (\ell(l_j) \text{ is the length of } l_j).
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Definition

A function \( f : [a, b] \to \mathbb{R} \) is said to be Henstock integrable if \( \exists A \) such that \( \forall \varepsilon > 0 \) there exists a gauge \( \delta \) on \( [a, b] \) such that if \( \dot{P} := \{(l_j, t_j)\}_{j=1}^n \) is \( \delta \)-fine, then

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|S(f; \dot{P}) - A| < \varepsilon \quad \text{(we say then } A \text{ is its integral).}
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McShane gave a surprising definition of the Lebesgue integral which goes in the same lines as the previous definition:

*If we do not require the tags \( t_j \) to belong to \( I_j \), but merely to \([a, b]\), then a miracle occurs! We obtain the Lebesgue integral.* See for example the book by Gordon or the one by McShane:

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In 1935 Denjoy studied the problem of integration of higher order differential coefficients.

Let $F$ be continuous on $[a, b]$, we say that $F$ has a Peano $n^{th}$ derivative at $x \in (a, b)$ if there are $n$ numbers $F_1(x), \ldots, F_n(x)$ such that

$$F(x + h) = F(x) + F_1(x)h + \cdots + F_n(x)\frac{h^n}{n!} + o(h^n), \quad \text{as } h \to 0.$$ 

We call each $F_j(x)$ its Peano $j^{th}$ derivative at $x$.

If $n > 1$ and this holds at every point, then $F'(x)$ exists everywhere, but this does not even imply that $F \in C^1[a, b]$. 
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Denjoy higher order integration problem

Suppose that $F$ has a Peano $n^{th}$ derivative $\forall x \in (a, b)$. Denjoy asked:

1. If $F_n(x) = 0$ for all $x \in [a, b]$, is $F$ a polynomial of degree at most $n - 1$?

2. Is it possible to reconstruct $F$, in a constructive manner, from the values $F_n(x)$?

Denjoy solved these two problems with an extremely difficult “totalization procedure” (once again involving transfinite induction).

- In 1957, Łojasiewicz found, using distribution theory, a more transparent solution to the first problem.
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Distributions and functions

We denote by $\mathcal{D}(\mathbb{R})$ the Schwartz space of compactly supported smooth functions. Its dual space $\mathcal{D}'(\mathbb{R})$ is the space of Schwartz distributions.

Distributions will be denoted by $f, g, \ldots$, while functions by $f, g, \ldots$.

It is well known that if $f$ is (Lebesgue) integrable over any compact, then it corresponds in a unique fashion to the distribution

$$\langle f(x), \psi(x) \rangle = \int_{-\infty}^{\infty} f(x)\psi(x)\,dx,$$

This also holds for the Denjoy-Perron-Henstock integral! We write $f \leftrightarrow f$ whenever there is a precise association between a function and a distribution.
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Schwartz definition of distributions does not consider pointwisely defined values. Inspired by Denjoy, Łojasiewicz defined the value of a distribution at a point.

**Definition**

A distribution $f \in D'(\mathbb{R})$ is said to have a value, $f(x)$, distributionally, at the point $x \in \mathbb{R}$, if there exist $n$ and a continuous function $F$ such that $F^{(n)} = f$ near $x$, $F \leftrightarrow F$, and $F$ has Peano $n^{th}$ derivative $F_n(x) = f(x)$ at the point.

Equivalently, $f(x)$ exists if and only if for every $\varphi \in D(\mathbb{R})$

$$\lim_{\varepsilon \to 0} \langle f(x + \varepsilon t), \varphi(t) \rangle = f(x) \int_{-\infty}^{\infty} \varphi(t) dt.$$
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Łojasiewicz uniqueness theorem

Łojasiewicz was able to show the following fundamental theorem:

Theorem

Let $f \in D'(\mathbb{R})$. If $f$ has point values everywhere in $(a, b)$ and if $f(x) = 0$, $\forall x \in (a, b)$, then $f = 0$ on $(a, b)$.

Corollary (Denjoy first problem)

If a continuous function $F$ has zero Peano $n^{th}$ derivative everywhere on $(a, b)$, then it is a polynomial of degree at most $n - 1$.

Proof: Define $f = F^{(n)} \in D'(\mathbb{R})$, where $F \leftrightarrow F$, then $f(x) = 0$, for all point in the interval, thus, $F^{(n)} = f = 0$ on the interval. So, $F$ has to be a polynomial with the right degree.
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Łojasiewicz functions and distributions

Łojasiewicz theorem gives a precise meaning to $f \leftrightarrow f$.

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Let $f \in \mathcal{D}'(\mathbb{R})$. It is said to be a Łojasiewicz distribution if $f(x)$ exists for all $x \in \mathbb{R}$.

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Let $f : [a, b] \to \mathbb{R}$ be a function. It is said to be a Łojasiewicz function if there exists a Łojasiewicz distribution $f \in \mathcal{D}'(\mathbb{R})$ such that $f(x) = f(x)$ for all $x \in [a, b]$.

- Łojasiewicz functions are not continuous, in general.
- They are Baire class 1 functions, and thus Darboux functions.
- Not all Lebesgue (locally) integrable function is a Łojasiewicz function.
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Let $f : [a, b] \to \mathbb{R}$ be a function. It is said to be a Łojasiewicz **function** if there exists a Łojasiewicz distribution $f \in D'(\mathbb{R})$ such that $f(x) = f(x)$ for all $x \in [a, b]$.

- Łojasiewicz functions are not continuous, in general.
- They are Baire class 1 functions, and thus Darboux functions.
- Not all Lebesgue (locally) integrable function is a Łojasiewicz function.
Łojasiewicz functions and distributions

Łojasiewicz theorem gives a precise meaning to $f \leftrightarrow f$.

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Our motivation is the construction of a new integral so that:

- It integrates every Łojasiewicz function.
- It extends the Denjoy-Perron-Henstock integral, and in particular that of Lebesgue.
- It solves Denjoy second problem on the integration of higher order differential coefficients in a constructive way (Łojasiewicz functions do not solve this problem).
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\( E'(\mathbb{R}) \) denotes the space of compactly supported distributions, the dual of \( E(\mathbb{R}) = C^\infty(\mathbb{R}) \).

Given \( \phi \in E(\mathbb{R}) \), we define the \( \phi \)-transform of \( f \in E'(\mathbb{R}) \) as the smooth function of two variables:

\[
F_\phi f(x, y) = (f * \check{\phi}_y)(x), \quad (x, y) \in \mathbb{H} = \mathbb{R} \times \mathbb{R}_+,
\]

where \( \check{\phi}_y(t) := \frac{1}{y} \phi \left( -\frac{t}{y} \right) \).

We will always assume that \( \phi \) is normalized, meaning

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Upper and lower values of the $\phi$-transform

If $x_0 \in \mathbb{R}$, denote by $C_{x_0, \theta}$ the cone in $\mathbb{H}$ starting at $x_0$ of angle $\theta$,

$$C_{x_0, \theta} = \{(x, t) \in \mathbb{H} : |x - x_0| \leq (\tan \theta)t \}.$$

If $f \in \mathcal{E}'(\mathbb{R})$, then the upper and lower angular values of its $\phi$–transform at $x_0$ are

$$f^+_{\phi, \theta}(x_0) = \limsup_{(x, t) \to (x_0, 0)} \sup_{(x, t) \in C_{x_0, \theta}} F_{\phi}f(x, t)$$

$$f^-_{\phi, \theta}(x_0) = \liminf_{(x, t) \to (x_0, 0)} \inf_{(x, t) \in C_{x_0, \theta}} F_{\phi}f(x, t).$$

For $\theta = 0$, we obtain the upper and lower radial limits of the $\phi$–transform.
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For $\theta = 0$, we obtain the upper and lower radial limits of the $\phi-$transform.
The class $T_0$ consists of all positive normalized functions $\phi \in \mathcal{E}(\mathbb{R})$ that satisfy the following condition:

$$\exists \alpha < -1 \text{ such that } \phi^{(k)}(x) = O\left(|x|^\alpha^{-k}\right) \quad \text{as } |x| \to \infty.$$ 

The class $T_1$ is the subclass of $T_0$ consisting of those functions that also satisfy

$$x\phi'(x) \leq 0 \quad \text{for all } x \in \mathbb{R}.$$
Classes of test functions

**Definition**

- The class $\mathcal{T}_0$ consists of all positive normalized functions $\phi \in \mathcal{E}(\mathbb{R})$ that satisfy the following condition:

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  \[ x\phi'(x) \leq 0 \quad \text{for all} \quad x \in \mathbb{R}. \]
Definition of major distributional pairs

A pair \((u, U)\) is called a major distributional pair for the function \(f\) if:

1. \(u \in \mathcal{E}' [a, b],\ U \in \mathcal{D}' (\mathbb{R}),\ \text{and}\ U' = u.\)

2. \(U\) is a Łojasiewicz distribution, with \(U(a) = 0.\)

3. There exist a set \(E,\) with \(|E| \leq \aleph_0,\) and a set of null Lebesgue measure \(Z,\ m(Z) = 0,\) such that for all \(x \in [a, b] \setminus Z\) and all \(\phi \in \mathcal{I}_0\) we have

\[
(u)_{\phi,0}^- (x) \geq f(x) ,
\]

while for \(x \in [a, b] \setminus E\) and all \(\phi \in \mathcal{I}_1\)

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(u)_{\phi,0}^- (x) > -\infty .
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Definition of minor distributional pairs

A pair \((v, V)\) is called a minor distributional pair for the function \(f\) if:

1. \(v \in \mathcal{E}' [a, b], V \in \mathcal{D}' (\mathbb{R}),\) and \(V' = v.\)

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3. There exist a set \(E,\) with \(|E| \leq \aleph_0,\) and a set of null Lebesgue measure \(Z, m (Z) = 0,\) such that for all \(x \in [a, b] \setminus Z\) and all \(\phi \in \mathcal{T}_0\) we have

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(v)_{\phi,0}^+ (x) \leq f(x),
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(v)_{\phi,0}^+ (x) < \infty.
\]
The distributional integral

Definition

A function $f : [a, b] \rightarrow \overline{\mathbb{R}}$ is called distributionally integrable if it has both major and minor distributional pairs and if

$$\sup_{(v,V)} V(b) = \inf_{(u,U)} U(b).$$

When this is the case this common value is the integral of $f$ over $[a, b]$ and is denoted as

$$(\text{dist}) \int_a^b f(x) \, dx,$$

or just as $\int_a^b f(x) \, dx$ if there is no risk of confusion.
Properties

We list some properties:

- Distributionally integrable functions are measurable and finite almost everywhere.
- Any Denjoy-Perron-Henstock integrable function is distributionally integrable, and the two integrals coincide within this class of functions.
- Any Łojasiewicz function is distributionally integrable, but not conversely.
- The distributional integral integrates higher order differential coefficients, and thus solves Denjoy’s second problem in a constructive manner.
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Assume $f$ is distributionally integrable on $[a, b]$ and set

$$F(x) := \int_a^x f(t)\,dt \quad x \in [a, b].$$

Then $F$ is a Łojasiewicz function. Moreover if $F \leftrightarrow F$, then $F'$ has distributional point values almost everywhere, and actually,

$$f(x) = F'(x), \quad a.e.$$
The integrals of Denjoy, Perron, and Henstock
From Denjoy to Łojasiewicz
The Distributional Integral

Indefinite integrals

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**Theorem**

Let \( f \) be distributionally integrable over \([a, b]\), let its indefinite integral be \( F \), with associated distribution \( F \), \( F \leftrightarrow F \), and let \( f = F' \in \mathcal{E}'(\mathbb{R}) \), so that \( f(x) = f(x) \) almost everywhere in \([a, b]\).

Then for any \( \psi \in \mathcal{E}(\mathbb{R}) \),

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$$\langle f, \psi \rangle = (\text{dist}) \int_{a}^{b} f(x) \psi(x) \, dx.$$
Given \( \{c_n\}_{n=1}^{\infty} \), define the function

\[
f(x) = \begin{cases} 
0, & \text{if } x \leq 0 \text{ or } x \geq 1, \\
c_n, & \text{if } \frac{1}{n+1} \leq x < \frac{1}{n}.
\end{cases}
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(1)

Let \( a_n = c_n \left(\frac{1}{n} - \frac{1}{n+1}\right) \), so that

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\int_{x}^{1} f(t) dt = \sum_{n \leq x^{-1}} a_n + c_{\lfloor 1/x \rfloor} \left(\frac{1}{[1/x]} - x\right), \quad x \in (0, 1).
\]

Then \( f \) is, on the interval \([0, 1]\),

- Lebesgue integrable if and only if \( \sum_{n=1}^{\infty} |a_n| < \infty \).
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(Continuation of last example)

In case $\sum_{n=1}^{\infty} a_n$ is Cesàro summable, we have

$$(\text{dist}) \int_{0}^{1} f(x) \, dx = \sum_{n=1}^{\infty} a_n \quad (C).$$

For example, if $c_n = (-1)^n n(n + 1)$, so that $a_n = (-1)^n$, we obtain

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and this function is not Denjoy-Perron-Henstock integrable.
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Example

Consider the functions $s_\alpha(x) := |x|^\alpha \sin(1/x)$ for $\alpha \in \mathbb{C}$. Near $x = 0$:

- If $\Re \alpha > -1$, then it is Lebesgue integrable.
- If $-1 \geq \Re \alpha > -2$, then it is not Lebesgue integrable but Denjoy-Perron-Henstock integrable.
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The family of distributions $s_\alpha$, where $s_\alpha \leftrightarrow s_\alpha$, is analytic in $\alpha$. 
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J.Vindas
A General Integral
Example

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The family of distributions $s_\alpha$, where $s_\alpha \leftrightarrow s_\alpha$, is analytic in $\alpha$. 
For further details about this new integral, I refer to my joint article with R. Estrada: