Factorization theorems in Denjoy-Carleman classes associated to representations of $(\mathbb{R}^d, +)$

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Introduction

Factorization theorems in modules over function algebras is an important subject with a long tradition in mathematical analysis.

A module $\mathcal{M}$ over a non-unital algebra $\mathcal{A}$ is said to have the strong factorization property if

$$\mathcal{M} = \mathcal{A} \cdot \mathcal{M} = \{a \cdot m \,|\, a \in \mathcal{A}, m \in \mathcal{M}\}.$$

It is said to have the weak factorization property if

$$\mathcal{M} = \operatorname{span}(\mathcal{A} \cdot \mathcal{M}).$$

We will present some new results about strong factorization:

1. A strong factorization theorem of Dixmier-Malliavin type for ultradifferentiable vectors of representations of $(\mathbb{R}^d, +)$.
2. We have established the strong factorization property for many families of convolution modules of ultradifferentiable functions. We will give some examples.

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Factorization in classical function algebras

- Factorization theorems on $\mathbb{T}$ go back to Salem and Zygmund.
- Rudin showed (1957-1958): $L^1(\mathbb{R}^d) = L^1(\mathbb{R}^d) * L^1(\mathbb{R}^d)$.
- Cohen (1959) extended this result to the function algebra of a locally compact group G,

$$L^1(G) = L^1(G) * L^1(G).$$

- Hewitt (1964) used Cohen technique to prove a general factorization theorem for Banach modules.
- Cohen-Hewitt type factorization theorems also hold for various Fréchet modules.
- Essential hypothesis: existence of bounded approximative units on the algebra under consideration.
- Many locally convex algebras do not have bounded approximative units. Examples: many algebras of smooth functions occurring in analysis.

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Factorization in algebras of smooth functions

- Ehrenpreis' problem (1960):

Does $\mathcal{D}(\mathbb{R}^d)$ factorize as $\mathcal{D}(\mathbb{R}^d) = \mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d)$?

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$$\mathcal{D}(\mathbb{R}^d) = \operatorname{span}(\mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d))$$

- If $d \geq 3$, they also showed that $\mathcal{D}(\mathbb{R}^d)$ does not have the strong factorization property.
- Dixmier and Malliavin (1979): negative answer for $d = 2$.
- Yulmukhametov (1999): in contrast $\mathcal{D}(\mathbb{R}) = \mathcal{D}(\mathbb{R}) * \mathcal{D}(\mathbb{R})$ holds.
- Several authors have independently shown (Miyazaki; Petzeltová and P. Vrbová; Dixmier and Malliavin; Voigt; ...)

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Factorization on Lie groups

- Let G be a real connected Lie group.
- Dixmier and Malliavin showed (1979) that

$$\mathcal{D}(G) = \text{span}(\mathcal{D}(G) * \mathcal{D}(G))$$

and, when additionally G is nilpotent,

$$\mathcal{S}(G) = \mathcal{S}(G) * \mathcal{S}(G).$$

(hereafter: convolution = left convolution)

- Let E be a locally convex Hausdorff space (lcHs) and denote as $\text{GL}(E)$ its group of isomorphisms.
- A group homomorphism $\pi : G \to \text{GL}(E)$ such that

$$G \times E \to E, \quad (g, e) \mapsto \pi(g)e$$

is separately continuous is a representation of G on E.

- We call $e \in E$ a smooth vector if its orbit mapping

$$G \to E \quad g \mapsto \pi(g)e, \quad \text{belongs to } C^\infty(G; E).$$

- E^∞ is the subspace of smooth vectors.

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Dixmier-Malliavin factorization theorems

- A representation of G on a sequentially complete lcHs E induces an action of the convolution algebra $\mathcal{D}(G)$ on the smooth vectors,

$$(f,e)\mapsto \Pi(f)e, \quad \mathcal{D}(G)\times E^{\infty}\to E^{\infty}, \quad \text{where}$$

$$\boxed{\Pi(f)e=\int_G f(g)\pi(g)e\,\mathrm{d}g\in E}$$

- If E is Banach and the representation is bounded, the action extends to $\mathcal{S}(G)$ and we can regard E^{∞} as a module over $\mathcal{S}(G)$.

Theorem

If E is a Fréchet space, E^{∞} has the weak factorization property w.r.t. $\mathcal{D}(G)$, that is, $\boxed{E^{\infty}=\operatorname{span}(\Pi(\mathcal{D}(G))E^{\infty})}$.

Theorem

If E is a Hilbert space, the representation is unitary, and G is nilpotent, then E^{∞} has the strong factorization property w.r.t. $\mathcal{S}(G)$.

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Analytic factorization of Lie group representations

- $e \in E$ is an analytic vector if $g \mapsto \pi(g)e$ is an analytic mapping.
- E^ω: subspace of analytic vectors.
- A representation is called an F-representation if
  - E is a Fréchet space;
  - there is a basis of continuous seminorms $(p_n)_{n\in\mathbb{N}}$ such that for each n the action $G \times (E, p_n) \to (E, p_n)$ is continuous.
- For F-representations, we get an action of the algebra of exponentially rapidly decreasing analytic functions $\mathcal{A}(G)$ on E^ω.

Theorem (Gimperlein, Krötz, and Lienau (2012))

For F-representations, E^ω has the weak factorization property w.r.t. $\mathcal{A}(G)$, that is, $\boxed{E^\omega = \operatorname{span}(\Pi(\mathcal{A}(G))E^\omega)}$.

Conjecture

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Analytic factorization for $(\mathbb{R}^d, +)$

The convolution algebra $\mathcal{A}(\mathbb{R}^d)$ consists of real analytic functions f admitting holomorphic extension to $\mathbb{R}^d + i]-h, h[^d$ for some $h > 0$ and satisfying

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For F-representations of $\mathbb{R}^d$, E^ω has the strong factorization property w.r.t. $\mathcal{A}(\mathbb{R}^d)$, that is, $\boxed{E^\omega = \Pi(\mathcal{A}(\mathbb{R}^d))E^\omega}$.

Our results hold for more general representations than F-representations (see our preprint for definitions):

- projective generalized proto-Banach representations;
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We implicitly assume below that all representations are of one of these two types.

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The convolution algebra $\mathcal{A}(\mathbb{R}^d)$ consists of real analytic functions f admitting holomorphic extension to $\mathbb{R}^d + i]-h, h[^d$ for some $h > 0$ and satisfying

$$\sup_{|\operatorname{Im} z| \leq h} e^{n|\operatorname{Re} z|}|f(z)| < \infty, \qquad \text{for each } n \in \mathbb{N}.$$

Theorem (Debrouwere, Prangoski, and V. (2020))

For F-representations of $\mathbb{R}^d$, E^ω has the strong factorization property w.r.t. $\mathcal{A}(\mathbb{R}^d)$, that is, $\boxed{E^\omega = \Pi(\mathcal{A}(\mathbb{R}^d))E^\omega}$.

Our results hold for more general representations than F-representations (see our preprint for definitions):

- projective generalized proto-Banach representations;
- inductive generalized proto-Banach representations.

We implicitly assume below that all representations are of one of these two types.

Denjoy-Carleman classes

Consider a log-convex sequence $M = (M_p)_p$ of positive numbers and set $m_p = M_p/M_{p-1}$. We assume:

$(M.2)$ there are $C_0, H > 0$ such that $M_{p+q} \leq C_0 H^{p+q} M_p M_q$;

$(M.2)^*$ there are $Q, p_0 \in \mathbb{N}$ such that $2m_p \leq m_{Qp}$ for all $p \geq p_0$.

Prototypical example: $M_p = (p!)^\sigma$, with $\sigma > 0$.

- a vector $e \in E$ is ultradifferentiable of class $[M]$ if its orbit mapping w.r.t. the representation is bornologically ultradifferentiable of class $[M]$.
- $[M]$ is the common notation for both the Beurling (M) and $\{M\}$ Roumieu cases of ultradifferentiability.
- $E^{[M]}$ denotes the space of ultradifferentiable vectors of class $[M]$ of a representation.
- Note that $E^\omega = E^{\{p!\}}$.

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Factorization theorem for ultradifferentiable vectors

For $h > 0$, we define the Fréchet space

$$\mathcal{K}^{M,h}(\mathbb{R}^d) = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \sup_{\alpha\in\mathbb{N}^d} \sup_{x\in\mathbb{R}^d} \frac{h^{|\alpha|}|\partial^\alpha\varphi(x)|e^{n|x|}}{M_{|\alpha|}} < \infty, \quad \forall n \in \mathbb{N}\}.$$

We set

$$\mathcal{K}^{(M)}(\mathbb{R}^d) = \varprojlim_{h\to\infty} \mathcal{K}^{M,h}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{K}^{\{M\}}(\mathbb{R}^d) = \varinjlim_{h\to 0^+} \mathcal{K}^{M,h}(\mathbb{R}^d).$$

If $M_p = p!$, then $\mathcal{A}(\mathbb{R}^d) = \mathcal{K}^{\{M\}}(\mathbb{R}^d)$.

Theorem (Debrouwere, Prangoski, and V. (2020))

Let (π, E) be either a projective or an inductive generalized proto-Banach representation of $(\mathbb{R}^d, +)$ on a sequentially complete lcHs E. Then, $E^{[M]}$ has the strong factorization property w.r.t. $\mathcal{K}^{[M]}(\mathbb{R}^d)$

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Factorization of modules of ultradifferentiable functions

- Our factorization theorem implies the strong factorization property for many concrete families of modules of ultradifferentiable functions.

Example:

- Let $\omega : \mathbb{R}^d \to (0, \infty)$ be a continuous weight function satisfying

$$\sup_{x \in \mathbb{R}^d} \frac{\omega(x + \cdot)}{\omega(x)} \in L^\infty_{loc}(\mathbb{R}^d).$$

- Consider $E = L^p_\omega = \{f |\, \omega \cdot f \in L^p(\mathbb{R})\}$ if $1 \leq p < \infty$.
- The ultradifferentiable vectors are (w.r.t. regular representation)

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References

For more details, see our preprint

- A. Debrouwere, B. Prangoski, J. Vindas, *Factorization in Denjoy-Carleman classes associated to representations of* $(\mathbb{R}^d, +)$, preprint, arXiv:2006.04861.

Related works on factorization theorems for representations:

- J. Dixmier, P. Malliavin, *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Sci. Math. **102** (1978), 307–330.
- H. Gimperlein, B. Krötz, C. Lienau, *Analytic factorization of Lie group representations*, J. Funct. Anal. **262** (2012), 667–681.