An introduction to hyperfunctions

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Hefei, China, July 14, 2026

1 Introduction: The goal of this talk is to give an elementary introduction to hyperfunction theory. Our material here is based on [3,4] (see also [1,2]).

Hyperfunction theory is an approach to the theory of generalized functions discovered by M. Sato.

In general, the goal of a generalized function theory is to construct a sheaf $\mathcal{F}$ of vector spaces on, say, $\mathbb{R}^n$ such that

① $C \subseteq \mathcal{F}$ as a subsheaf (here C is the sheaf of continuous functions)

② There are sheaf operators $\partial_1, \dots, \partial_n : \mathcal{F} \to \mathcal{F}$ such that $\partial_j : C^1 \to C$ are just the partial derivatives:

③ There is a (multiplication) subalgebra $R \subseteq C^{\infty}$ (regular elements) such that $\mathcal{Y}$ is a "natural" R-module.

The theory of Schwartz distributions solves these problems with $R = C^{\infty}$ and $\mathcal{Y} = \mathcal{D}'$ the sheaf of distributions.

Sato obtained an extension of Schwartz's theory, the sheaf of hyperfunctions $\mathcal{B}$ which solves the problem with $R = \mathcal{A}$ the sheaf of real analytic functions. We consider here hyperfunctions of one-variable in some detail, and sketch an approach to the theory in several variables. In what follows, we denote by $\mathcal{O}(W)$ the space of holomorphic functions on an open set W.

2 Some tools from complex analysis in one variable

Our first result is the following Mittag-Leffler theorem.

<u>Theorem 2.1</u> (Mittag-Leffler) Let $W \subseteq \mathbb{C}$ be open and let $\{W_\alpha\}_{\alpha \in I}$ be a covering of W.

Suppose $\varphi_{\alpha,\beta} \in \mathcal{O}(W_\alpha \cap W_\beta)$ are given such that

$$\varphi_{\alpha,\beta} + \varphi_{\beta,\gamma} + \varphi_{\gamma,\alpha} = 0, \quad \alpha,\beta,\gamma \in I.$$

Then, there is $\{\varphi_\alpha\}_{\alpha \in I}$ with $\varphi_\alpha \in \mathcal{O}(W_\alpha)$ such that

$$\varphi_{\alpha,\beta} = \varphi_\beta - \varphi_\alpha \text{ on } W_\alpha \cap W_\beta$$

Remark 2.2 This theorem says that the 1st cohomology group on W with coefficients in $\mathcal{O}$ vanishes; in symbols: $H^1(W;\mathcal{O}) = 0$.

The next tool is the representations of analytic functionals.

Definition 2.3. Let $K \subseteq \mathbb{C}$ be compact and W be open.

(1) The space of germs of holomorphic functions on K is

$$\mathcal{O}(K) = \varinjlim_{K \subseteq V} \mathcal{O}(V).$$

Its dual space $\mathcal{O}'(K)$ is the space of analytic functionals on K. Note that $\mathcal{O}(K)$ is a DFS-space (dual Fréchet-Schwartz space), while $\mathcal{O}'(K)$ is a Fréchet space.

(2) We write $H^1_K(W;\mathcal{O}) = \mathcal{O}(W \setminus K)/\mathcal{O}(W)$.

Theorem 2.4 (Sebastião e Silva–Köthe–Grothendieck). Let $K \subseteq W \subseteq \mathbb{C}$ with K compact and W open. Then

$$(2.1) \quad \mathcal{O}'(K) \cong H^1_K(W;\mathcal{O})$$

The isomorphism $H^1_K(W;\mathcal{O}) \longrightarrow \mathcal{O}'(K)$ is generated by the bilinear form $\langle [F], f \rangle$ on $H^1_K(W;\mathcal{O}) \times \mathcal{O}(K)$:

$$(2.2) \quad \langle [F], f \rangle = -\int_\gamma F(z) f(z)\, dz,$$

where $F \in \mathcal{O}(W \setminus K)$, $f \in \mathcal{O}(U)$, $K \subseteq U \subseteq W$, and γ is a contour in U winding once around K.

Conversely, given $\mu \in \mathcal{O}'(K)$, μ can be represented as in (2.2) with F given by its Cauchy transform:

$$F(z) = \frac{1}{2\pi i} \left\langle \mu(\zeta), \frac{1}{\zeta - z} \right\rangle, \quad z \in \mathbb{C} \setminus K.$$

Remark 2.5 The equality (2.1) shows that $H^1_K(W;\mathcal{O})$ is independent of W.

If W and W' are neighborhoods of K s.t. $W' \subseteq W$ we have (they are exact if every component of W intersects $W \setminus K$ (resp. W'))

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}(W) \arrow[r] \arrow[d] & \mathcal{O}(W\setminus K) \arrow[r] \arrow[d] & H^1_K(W;\mathcal{O}) \arrow[r] \arrow[d, "(\ )"] & 0 \\ 0 \arrow[r] & \mathcal{O}(W') \arrow[r] & \mathcal{O}(W'\setminus K) \arrow[r] & H^1_K(W';\mathcal{O}) \arrow[r] & 0 \end{tikzcd}$$

so that $H^1_K(W;\mathcal{O}) \to H^1_K(W';\mathcal{O})$ can be defined to make the diagram commutative. (call restriction map).

Corollary 2.6 $H^1_K(W;\mathcal{O}) \to H^1_K(W';\mathcal{O})$ is a linear topological isomorphism.

Definition 2.7 $$H^1[K;\mathcal{O}] = \varinjlim_{K \subseteq W} H^1_K(W;\mathcal{O}) = \varinjlim_{K \subseteq W} \mathcal{O}(W\setminus K)/\mathcal{O}(W)$$

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3 Sato's hyperfunctions in one-variable.

Sato's idea was to consider boundary values of analytic functions on $\mathbb{R}$ to construct the space of hyperfunctions. The precise definition of Sato's sheaf of hyperfunctions is as follows.

Let A be s.t. $A \subseteq W$ and $A \subseteq W'$ are closed in both open sets W and W' of $\mathbb{C}$.

We write $H^1_A(W;\mathcal{O}) = \mathcal{O}(W \setminus A)/\mathcal{O}(W)$

if every connected component of W meets $W \setminus A$ then

$$0 \longrightarrow \mathcal{O}(W) \longrightarrow \mathcal{O}(W \setminus A) \to H^1_A(W;\mathcal{O}) \to 0$$

is a short exact sequence. As before

$$\begin{tikzcd} \mathcal{O}(W) \arrow[r] \arrow[d] & \mathcal{O}(W \setminus A) \arrow[d] \\ \mathcal{O}(W') \arrow[r] & \mathcal{O}(W' \setminus A) \end{tikzcd}$$

induces a canonical homomorphism

$$(3.1) \qquad H^1_A(W;\mathcal{O}) \longrightarrow H^1_A(W';\mathcal{O})$$

which one checks easily to be injective.

<u>Theorem 3.1</u> Let $A \subseteq W' \subseteq W$ be as above. Then (3.1) is an isomorphism.

<u>Proof.</u> We will check the surjectivity. Let $\varphi \in \mathcal{O}(W' \setminus A)$

We find $\psi \in \mathcal{O}(W \setminus A)$ s.t. $[\psi] = [\varphi]$ in $\mathcal{O}(W' \setminus A)/\mathcal{O}(W')$. For it, we apply the Mittag-Leffler theorem to $\varphi \in \mathcal{O}(W' \setminus A)$ with $(W \setminus A) \cup W'$ so that $(W \setminus A) \cap W' = W' \setminus A$, so that $\exists \psi \in \mathcal{O}(W \setminus A)$ and $\eta \in \mathcal{O}(W')$ s.t. $\varphi = \psi - \eta$ on $W' \setminus A$. ////

Def 3.2 Let $S \subseteq \mathbb{R}$ be locally closed. We write

$$\mathcal{B}[S] = H^1[S; \mathcal{O}] = \varinjlim_{W \in \mathcal{N}(S)} H^1_S[W; \mathcal{O}],$$

where $\mathcal{N}(S)$ is the collection of complex neighborhoods of S for which S is open in them.

If $S = I$ is open, we write $\mathcal{B}(I) = \mathcal{B}[I]$ and call $\mathcal{B}(I)$ the space of hyperfunctions on I.

Examples 3.3. (i) $\mathcal{B}(\mathbb{R}) = \mathcal{O}(\mathbb{C} \setminus \mathbb{R}) / \mathcal{O}(\mathbb{C})$

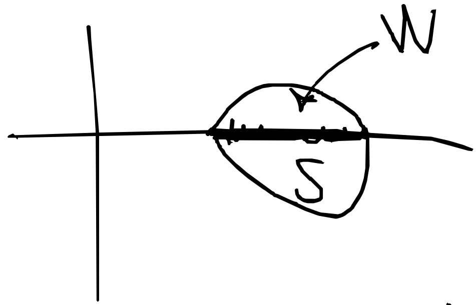
(ii) $\mathcal{B}[K] = \mathcal{O}'(K)$, the space of analytic functionals on a compact $K \subseteq \mathbb{R}$. ////

We note that $f \in \mathcal{B}[S]$ is the equivalence class of $\varphi \in \mathcal{O}(W \setminus S)$, where $W \in \mathcal{N}(S)$.

which we write

$$f=[\varphi] \quad \text{or} \quad f(x)=[\varphi(z)]_{z=x}$$

and called φ a defining function of f. Note $f=[\varphi]$ is a zero hyperfunction if $\varphi \in \mathcal{O}(W)$.



<u>Proposition</u> 3.4 Let $S \subseteq \mathbb{R}$ be locally closed and I be a real neighborhood of S. The natural mapping

$$\iota : \mathcal{B}[S] \longrightarrow \mathcal{B}(I)$$

is injective.

<u>Proof.</u> $W \in \mathcal{N}(S)$ s.t. $W \cap \mathbb{R} = I$.
Take $\varphi \in \mathcal{O}(W \setminus S)$ s.t. $f=[\varphi]$ on $\mathcal{B}[S]$.

$$\iota f = [\varphi|_{W \setminus I}]$$

is clearly injective. ///

We set $\mathbb{C}_{\pm} = \{ z \in \mathbb{C} : \pm \operatorname{Im} z > 0 \}$

If W is open in $\mathbb{C}$, $W^{\pm} = W \cap \mathbb{C}^{\pm}$.

If $W \in \mathcal{N}(I)$, I open in $\mathbb{R}$ s.t. $W \cap \mathbb{R} = I$

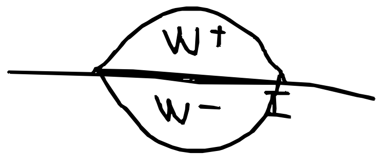
Then

$$\mathcal{O}(W \setminus I) = \mathcal{O}(W^+) \oplus \mathcal{O}(W^-)$$

so that each $\varphi = \varphi_+ + \varphi_-$ with.

$$\varphi_+(z) = \begin{cases} \varphi(z), & z \in W^+ \\ 0, & z \in W^- \end{cases}, \quad \varphi_-(z) = \begin{cases} 0, & z \in W^+ \\ \varphi(z), & z \in W^- \end{cases}$$

We have $[\varphi] = 0$ iff φ_+ and φ_- are analytic continuation of each other across I.



We define its boundary values $\varphi(x \pm i0) \in \mathcal{B}(I)$

by
$$\varphi(x+i0) = [\varphi_+(z)]_{z=x},$$
$$\varphi(x-i0) = -[\varphi_-(z)]_{z=x}.$$

So that,

$$f(x) = [\varphi(x)]_{z=x} = \varphi(x+i0) - \varphi(x-i0).$$

Recall $f \in \mathcal{A}(I)$ (f is real analytic on $I \subseteq \mathbb{R}$)

if there is $W \in \mathcal{N}(I)$ such that f extends holomorphically to W.

<u>Proposition 3.5</u> The association of $f \in \mathcal{A}(I)$ with the hyperfunction

$$f(x) := f(x+i0) = f(x-i0) \in \mathcal{B}(I),$$

defines an injection $\mathcal{A}(I) \longrightarrow \mathcal{B}(I)$, so that we regard $\mathcal{A}(I) \subseteq \mathcal{B}(I)$. We have $\mathcal{B}(I)$ is an $\mathcal{A}(I)$-module.///

<u>Proof</u> If $f(x+i0) = 0$ it means $\begin{cases} f(z), & z \in W^+ \\ 0, & z \in W^- \end{cases}$ has analytic continuation across I, so that $f = 0$.

Note that $f(x+i0) = f(x-i0) \Leftrightarrow f(x+i0) - f(x-i0) = 0$ and the latter trivially holds.///

If $S' \subset S$ is open in a locally closed set S given $W \in \mathcal{N}(S)$ $\exists\, W' \in \mathcal{N}(S')$ s.t. $W' \setminus S' \subseteq W \setminus S$ (E.g. $W' = (W \setminus S) \cup S'$)

Then.

we obtain a canonical homomorphism

$$H^1_S(W;\mathcal{O}) \longrightarrow H^1_{S'}(W';\mathcal{O})$$

which induces (by definition) the restriction mapping for hyperfunctions

$$\mathcal{B}[S] \longrightarrow \mathcal{B}[S'].$$

Theorem 3.6 Let $I \subseteq \mathbb{R}$ be open. $\mathcal{B}$ is a sheaf on I. Concretely, if $\{I_\alpha\}_{\alpha\in J}$ is an open covering of I, we have:

① If $f \in \mathcal{B}(I)$ satisfies $f\restriction_{I_\alpha}=0$, $\forall \alpha \in J$, then $f=0$.

② Given $f_\alpha \in \mathcal{B}(I_\alpha)$ for each $\alpha \in J$ s.t.

$$f_\alpha|_{I_\alpha \cap I_\beta} = f_\beta|_{I_\alpha\cap I_\beta}, \quad \forall \alpha,\beta \in J,$$

there is $f \in \mathcal{B}(I)$ s.t. $f|_{I_\alpha} = f_\alpha$.

Proof. ① Clear. ② Take $W_\alpha \in \mathcal{N}(I_\alpha)$ s.t. $W_\alpha \cap \mathbb{R} = I_\alpha$ and $\varphi_\alpha \in \mathcal{O}(W_\alpha \setminus I_\alpha)$ s.t. $f_\alpha = [\varphi_\alpha]$. Then

$$W = \bigcup W_\alpha \in \mathcal{N}(I) \text{ and } W \cap \mathbb{R} = I.$$

Our assumption is that $\varphi_{\alpha,\beta} = \varphi_\beta - \varphi_\alpha \in \mathcal{O}(W_\alpha \cap W_\beta)$. We can therefore, using the Mittag-Leffler theorem, find $\psi_\alpha \in \mathcal{O}(W_\alpha)$ s.t. $\varphi_{\alpha,\beta} = \psi_\alpha - \psi_\beta$ on $W_\alpha \cap W_\beta$. This says $\varphi_\alpha - \psi_\alpha = \varphi_\beta - \psi_\beta$ on $(W_\alpha \setminus I_\alpha) \cap (W_\beta \setminus I_\beta)$. We can define $\varphi \in \mathcal{O}(W \setminus I) = \mathcal{O}(\bigcup W_\alpha \setminus I_\alpha)$ by defining $\varphi = \varphi_\alpha - \psi_\alpha$ on $W_\alpha \setminus I_\alpha$.

Then $f = [\varphi] \in \mathcal{B}(I)$ satisfies

$$f|_{I_\alpha} = [\varphi_\alpha] - [\psi_\alpha] = f_\alpha - 0 = f_\alpha. \quad ///$$

The following property of the sheaf of hyperfunctions is very important.

Def 3.7. A sheaf $\mathcal{F}$ over a topological space X is called flabby if for each open $U \subseteq X$ and each $s \in \mathcal{F}(U)$ $\exists g \in \mathcal{F}(X)$ s.t. $g|_U = s$ ///

Theorem 3.8 The sheaf of hyperfunctions $\mathcal{B}$ is flabby.

Proof. Let $I \subseteq$ open. We consider $\partial I = \overline{I} \setminus I$

Then $W = \mathbb{C} \setminus \partial I \in \mathcal{N}(I)$ and $W \supseteq \mathbb{C} \setminus \mathbb{R}$.

We take a defining function $\varphi \in \mathcal{O}((\mathbb{C} \setminus \partial I) \setminus I)$

$= \mathcal{O}(\mathbb{C} \setminus \overline{I})$.

of $f \in \mathcal{B}(I)$. Now $g = [\varphi|_{\mathbb{C} \setminus \mathbb{R}}] \in \mathcal{B}(\mathbb{R})$

satisfies $g|_I = f$ //

As for any sheaf we can define supp f as the complement of the largest open set I where $f|_I = 0$.

Corollary 3.9 Let $I \subseteq \mathbb{R}$ open and S be closed in I.

Then,

$$\mathcal{B}[S] \cong \{ f \in \mathcal{B}(I) : \operatorname{supp} f \subseteq S \}$$

Proof. Let $J = I \setminus S$ an open subset. By Theorem 3.8 the restriction mapping in

$$0 \longrightarrow \mathcal{B}[I \setminus J] \xrightarrow{\text{canonical inclusion}} \mathcal{B}(I) \xrightarrow{\text{restriction}} \mathcal{B}(J) \to 0$$

is surjective. Thus this short sequence is exact, implying $\mathcal{B}[S] = \mathcal{B}[I \setminus J] \cong \{ f \in \mathcal{B}(I) : f|_J = 0 \}$

We can define differentiation in an easy way.

$$\frac{df}{dx} = [\varphi'(z)]_{z=x}, \quad \text{where } f = [\varphi].$$

To embed continuous functions (or even distributions) when they are compactly supported, we can employ the Cauchy transform. In fact, notice first that if $f \in C(\mathbb{R})$ and $\operatorname{supp} f \subseteq [a,b]$, then

$$F(z) = -\frac{1}{2\pi i}\int_a^b \frac{f(u)}{u-z}\,du, \quad z \notin [a,b]$$

belongs to $\mathcal{O}(\mathbb{C}\setminus[a,b])$. The assignment $f \mapsto [F]$ is injective because ($y>0$):

$$F(x+iy) - F(x-iy) = \frac{1}{2\pi i}\int_a^b f(u)\left[\frac{1}{u-x-iy} - \frac{1}{u-x+iy}\right]du$$

$$= \frac{1}{\pi y}\int_a^b f(u)\,\frac{1}{1+\left(\frac{u-x}{y}\right)^2}\,du \xrightarrow{y\to 0^+} f(x).$$

The embedding of $C(I)$ or more generally the space of distributions $\mathcal{D}'(I)$

can then be achieved via the following general result. A sheaf on a topological space X is called soft if every section on a closed set can be extended to a global section on X. Any flabby sheaf is soft, so $\mathcal{B}$ is soft. Sheaves admitting partitions of unity (the so-called fine sheaves) are also soft. The sheaf of continuous functions $\mathcal{C}$ and of distributions $\mathcal{D}'$ are soft.

<u>Lemma 3.10</u> Let X be a second countable topological space and $\mathcal{F}$ and $\mathcal{G}$ be soft sheaves on X. Suppose there is a linear mapping

$$\iota_c : \mathcal{F}_c(X) = \{ s \in \mathcal{F}(X) : \operatorname{supp} s \text{ is compact} \} \longrightarrow \mathcal{G}_c(X)$$

such that $\operatorname{supp} \iota_c(s) = \operatorname{supp} s, \ \forall s \in \mathcal{F}_c(X)$.

Then there is a unique sheaf monomorphism $\rho : \mathcal{F} \longrightarrow \mathcal{G}$ such that for every open $U \subseteq X$, one has $C_U(S) = C_c(S)$ $\forall S \in \mathcal{H}_c(U)$. //

<u>Corollary</u> We have sheaf embeddings

$$\mathcal{C} \longrightarrow \mathcal{D}' \longrightarrow \mathcal{B}.$$

On compactly supported distributions the embedding is induced by the Cauchy transform.

$\boxed{4}$ Hyperfunctions in several variables.

We have seen that hyperfunctions in one variable satisfy:

1. $I \longrightarrow \mathcal{B}(I)$ is a sheaf over $\mathbb{R}$.
2. $\mathcal{B}$ is a flabby sheaf.
3. Its compact sections are the space of analytic functionals: $\mathcal{B}[K] = \mathcal{O}'(K) = \mathcal{A}'(K)$, $K \subseteq$ compact $\mathbb{R}$.

These three properties essentially characterize $\mathcal{B}$ in view of Lemma 3.10.

Hyperfunctions in several variables satisfy these same properties. However, the original construction by Sato was not easy to achieve.

If $\Omega \subseteq_{open} \mathbb{R}^n$, Sato defined

$$\mathcal{B}(\Omega) = H^n(V, V \setminus \Omega; \mathcal{O}),$$

where $V \subseteq \mathbb{C}^n$ is an open set such that $V \cap \mathbb{R}^n = \Omega$ and the right hand side is the nth <u>relative</u> cohomology group of the open pair $(V, V \setminus \Omega)$ with coefficients in the sheaf $\mathcal{O}$ of holomorphic functions, a concept he invented[1] to be able to construct the theory of hyperfunctions of several variables.

In [2], L. Hörmander gave a presentation of the sheaf of hyperfunctions using harmonic functions. These approach has the advantage of being similar to the one-dimensional theory. A clear exposition in this respect can be found in H. Komatsu's paper [3]. Here we sketch the presentation from [3].

Let $V \subseteq \mathbb{R}^{n+1}$ be open. We denote the space of

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[1] The same concept was independently defined by Grothendieck, called by him the nth local cohomology group with support in Ω and denoted $H^n_\Omega(V, \mathcal{O})$

harmonic functions on V as

$$\mathcal{H}(V) = \{\varphi \in C^\infty(V) : \Delta\varphi = 0\}.$$

We write variables in $\mathbb{R}^{n+1}$ as $(x,t) \in \mathbb{R}^n \times \mathbb{R}$.

If V is symmetric with respect to $\mathbb{R}^n$ that is

$$(x,t) \in V \Leftrightarrow (x,-t) \in V,$$

we write $\mathcal{H}_-(V)$ for the space of harmonic functions that are odd with respect to the variable t, that is,

$$\mathcal{H}_-(V) = \{\varphi \in \mathcal{H}(V) : \varphi(x,-t) = -\varphi(x,t)\}.$$

Given $\Omega \subseteq_{\text{open}} \mathbb{R}^n$, let $V \subseteq \mathbb{R}^{n+1}$ be a symmetric open set such that $V \cap \mathbb{R}^n = \Omega$. We define the space of hyperfunctions on Ω as

$$\mathcal{B}(\Omega) = \mathcal{H}_-(V \setminus \Omega) / \mathcal{H}_-(V)$$

If $\Omega_1 \subseteq \Omega$, one has natural restriction maps

$\mathcal{B}(\Omega) \longrightarrow \mathcal{B}(\Omega_1)$, making $\mathcal{B}$ a presheaf. One can show:

1. $\mathcal{B}$ is a sheaf. 2. $\mathcal{B}$ is flabby

3. If $\mathcal{B}[K] = \{f \in \mathcal{B}(\mathbb{R}^n) : \operatorname{supp} f \subseteq K\}$, then

$$\mathcal{B}[K] \cong \mathcal{A}'(K), \tag{4.1}$$

the Fréchet space of analytic functionals with support on K.

In addition, the definition of $\mathcal{B}(\Omega)$ does not depend on the choice of V. The latter and property ① follow from the following Mittag-Leffler theorem for harmonic functions shown by Malgrange.

The flabbiness can be shown as in Section 3.

<u>Theorem 4.1</u> Let $\{V_\alpha\}_{\alpha \in J}$ be open subsets of $\mathbb{R}^{n+1}$. If a family $\varphi_{\alpha,\beta} \in \mathcal{H}(V_\alpha \cap V_\beta)$ satisfies

$$\varphi_{\alpha,\beta} + \varphi_{\beta,\gamma} + \varphi_{\gamma,\alpha} = 0,$$

there are $\varphi_\alpha \in \mathcal{H}(V_\alpha)$ s.t. $\varphi_{\alpha,\beta} = \varphi_\beta - \varphi_\alpha$ on $V_\alpha \cap V_\beta$. ////

In other words, $H^1(V, \mathcal{H}) = 0$.

Given $f = [F] \in \mathcal{B}(\Omega)$, we simply write

$$f(x) = F(x,+0) - F(x,-0) = 2F(x,+0).$$

Suppose that g is real analytic in a neighborhood of a compact $K \subseteq \mathbb{R}^n$ and $f(x) = 2F(x,+0) \in \mathcal{B}[K]$.

We define the pairing $\langle f, g \rangle$

defining the isomorphism (4.1) as follows.
Let V be a symmetric neighborhood of K in $\mathbb{R}^{n+1}$ and $G \in H(V)$ s.t. it solves the Cauchy problem

$$\begin{cases} \Delta G = 0 \\ G(x,0) = 0 \\ \partial_t G(x,0) = g(x) \end{cases}$$

We may assume $F \in H_-(V \setminus K)$. We define

$$\langle f, g \rangle = -\int_{\mathbb{R}^{d+1}} F(x,t)\, \Delta(\rho G)(x,t)\, dx\, dt,$$

where ρ is a smooth function with compact support in V that is equal to 1 in a neighborhood of K.

Conversely, given an analytic functional $\mu \in A'(K)$, one defines the harmonic function given by the Poisson transform

$$\varphi(x,t) = \langle \mu(y), P(x-y,t) \rangle, \quad (x,t) \in \mathbb{R}^{n+1} \setminus K,$$

where $P(x,t) = \frac{1}{\omega_{n+1}} \frac{t}{(x^2+t^2)^{\frac{n+1}{2}}}$. Then $f = [\varphi]$ gives the inverse correspondence of the isomorphism (4.1). See [2,3] for details.

We refer to [1] for an extension of these ideas to quasianalytic functionals.

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