On the number of integers with number of prime factors in a given residue class.

(Ghent, November 6, 2019)

1 Introduction:

In classical number theory there is a number of asymptotic relations that become "elementarily equivalent" to the prime number theorem in the form:

$$(1) \qquad \pi(x) = \sum_{p \le x} 1 \sim \frac{x}{\log x}.$$

Two of them involve the Möbius function

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ has } k \text{ distinct prime factors} \\ 0 & \text{otherwise} \end{cases}$$

and the Liouville function $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ denotes the number of prime factors of n (counting multiplicities). These relations are

$$(2) \qquad M(x) = \sum_{n \le x} \mu(n) = o(x)$$

and

$$(3) \qquad L(x) = \sum_{n \le x} \lambda(n) = o(x).$$

The relation (3) basically says that asymptotically there are the same number of integers having a number of even prime factors as those having an odd number of them.

In a recent work Gregory Debruyne investigated a counterpart of the previous statement for arbitrary residue classes. In fact, let $K \geq 2$ and fix $c \in \{0, 1, \ldots, K-1\}$. Consider

(4) $$S_{K,c}(x) = \sum_{\substack{n \leq x \\ \Omega(n) \equiv c \bmod K}} 1 \; .$$

He has shown:

<u>Theorem 1</u> (Debruyne, 2019)

(5) $$S_{K,c}(x) \sim \frac{x}{K} \, . \quad ///$$

His result is actually more general and applies to Beurling generalized number systems. The proof he gave is based upon recent extensions of the Halász mean-value theorem due to him, F. Maes, and myself. Here we discuss a proof of the result for the integers as stated in Theorem 1.

$\boxed{2}$ Mean value theorems for multiplicative functions

Our proof of Theorem 1 is via mean value theorems, which we recall in this section.

Theorem 2 (Halász-Wirsing) Let f be real-valued and multiplicative with $|f(n)|\le 1$ then its mean-value exists and

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x} f(n) = \prod_p^{\infty}\left(1-\frac{1}{p}\right)\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\dots\right).$$

$$= \prod_p^{\infty}\left(1+\frac{f(p)-1}{p}+\frac{f(p^2)-f(p)}{p^2}+\dots\right)$$

Remark 1: If $\sum_p \frac{1-f(p)}{p}$ converges, then the latter product converges absolutely to a positive number. Otherwise $\sum_p \frac{1-f(p)}{p}=\infty$, and the logarithm of the product diverges to $-\infty$, i.e., in this case f has mean value 0,

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x} f(n)=0.$$

Corollary 1 $L(x) = \sum_{n\le x}\lambda(n) = o(x)$.

Proof. $\lambda(p) = -1$. So $\sum_p \frac{1-\lambda(p)}{p} = 2\sum_p \frac{1}{p} = \infty$, by Euler's theorem. ////

There is also a version of Theorem 2 for complex-valued multiplicative functions, due to Halász.

Theorem 3. (Halász) Let f be a multiplicative arithmetic function with $|f(n)|\le 1$. Then there are $c\in\mathbb{R}$, $\alpha\in\mathbb{R}$, and a slowly varying function ℓ, with $|\ell(x)|=1$, such that

$$(6)\qquad \sum_{n\le x} f(n) = c\cdot\frac{x^{1+i\alpha}}{1+i\alpha}\,\ell(\log x),\quad x\to\infty.$$

More precisely,

- If $\sum \frac{1-\mathrm{Re}(f(p)p^{-i\alpha})}{p}$ converges for some α,

$$\sum_{n\le x} f(n) = \frac{x^{1+i\alpha}}{1+i\alpha}\prod_{p\le x}\left(1-\frac{1}{p}\right)\left(1+\frac{f(p)}{p^{1+i\alpha}}+\frac{f(p^2)}{p^{2(1+i\alpha)}}+\cdots\right)+o(x)$$

• Otherwise, $\lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} f(n) = 0$. ///

We are interested in the otherwise case, which we summarize as follows:

<u>Corollary 2</u> $\sum_{n\le x} f(n) = o(x) \iff \sum_{p} \frac{1-\mathrm{Re}(p^{-i\alpha} f(p))}{p} = \infty$

$\forall \alpha \in \mathbb{R}$. ///

3 Prime number lemmas.

In this section we collect some results on prime numbers that we will be employing for the proof of Theorem 1. We use here the PNT in the form:

(7) $$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + o\left(\frac{x}{\log^2 x}\right).$$

<u>Lemma 1</u> For some $c \in \mathbb{R}$,

(8) $$\sum_{p\le x} \frac{1}{p} = \log\log x + c + O\left(\frac{1}{\log x}\right).$$ ///

<u>Proof.</u> The right hand side of (8) is

$$\int_1^x \frac{d\pi(u)}{u} = \frac{\pi(x)}{x} + \int_2^x \frac{\pi(u)}{u^2}\,du,$$ the rest

follows by substituting here (7) after a small computation. //

Remark 2: (8) can actually shown to be "elementarily" equivalent to the PNT in its weakest form

$$\pi(x) \sim \frac{x}{\log x},$$ passing through the so-called strong Mertens relation. ///

In the rest of this section we are interested in the sums

$$\sum_p \frac{1-\cos(\beta+\alpha\log p)}{p}. \tag{9}$$

Our goal is to show:

Proposition 1: For any $\beta \in \mathbb{R}$ and $\alpha \neq 0$, the sum (9) diverges to ∞.

Proof. Call (8) $T(x)$, so the partial sums of the cosine part of (9) are:

$$-\int_{2^-}^{x} \cos(\beta+\alpha\log u)\, dT(u) = -T(u)\cos(\beta+\alpha\log u)$$

$$- \int_{2^-} \frac{\sin(\beta+\alpha\log u)}{u} T(u)\, du$$

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$$= -\left(\log\log x + c + O\left(\frac{1}{\log x}\right)\right)\cos(\beta + \alpha\log x)$$

$$-\alpha\int_2^x \frac{\sin(\beta+\alpha\log u)}{u}\log\log u\, du + c\left[\cos(\beta+\alpha\log x)\right]\Big|_2^x$$

$$+ O\left(\int_2^x \frac{du}{u\log u}\right)$$

$$= \int_2^x \frac{\cos(\beta+\alpha\log u)\, du}{u\log u} + O(\log x).$$

$$= \int_2^{\log x} \frac{\cos(\beta+\alpha y)}{y}\, dy + O(\log x)$$

$$= \frac{1}{\alpha}\int_2^{\log x} \frac{\sin(\beta+\alpha y)}{y^2}\, dy + O(\log x) = O(\log x).$$

<u>Conclusion</u> (by Lemma 1 again!):

$$\sum_{p\le x} \frac{1-\cos(\beta+\alpha\log p)}{p} \sim \log\log x \longrightarrow \infty \quad ///$$

$\boxed{3}$ Back to Theorem 1.

Our starting point is the orthogonality relation in $\mathbb{Z}_K = \mathbb{Z}/K\mathbb{Z}$:

$$\frac{1}{K}\sum_{q=0}^{K-1} e^{\frac{2\pi i q \nu}{K}} e^{\frac{-2\pi i c q}{K}} = \begin{cases} 1 & \nu \equiv c \bmod K \\ 0 & \text{otherwise} \end{cases}$$

Now $S_{K,c}(x) = \sum_{n \leq x} h(n)$, where h is the characteristic function of $\{n : \Omega(n) \equiv c \bmod K\}$

The orthogonality relations then say,

$$S_{K,c}(x) = \frac{1}{K}\sum_{n\leq x}\sum_{q=0}^{K-1} e^{\frac{-2\pi i c q}{K}} \cdot e^{\frac{2\pi i \Omega(n)\cdot q}{K}}$$

$$= \frac{1}{K}\lfloor x \rfloor + \sum_{q=1}^{K-1} e^{\frac{-2\pi i c q}{K}} \cdot \sum_{n \leq x} e^{\frac{2\pi i \Omega(n) q}{K}}.$$

Therefore, if $f_q(n) = e^{\frac{2\pi i \Omega(n) q}{K}}$, it suffices to show that for each $q = 1, \ldots, K-1$,

$$\sum_{n \leq x} f_q(n) = o(x).$$

By Corollary 2, the latter holds iff for all $\alpha \neq 0$

$$\infty = \sum_p \frac{1 - \mathrm{Re}\left(f_q(p)\, p^{i\alpha}\right)}{p}$$

$$= \sum_p \frac{1 - \mathrm{Re}\left(e^{2\pi i \frac{q}{k}} \cdot e^{2i\alpha \log p}\right)}{p}$$

$$= \sum_p \frac{1 - \cos\left(\frac{2\pi q}{k} + \alpha \log p\right)}{p},$$

which is in fact the case in view of Proposition 1.

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