On Diamond-type error terms for Beurling integers

1 Introduction. (Ghent, March 15, 2022)

A (Beurling) generalized number system is an bounded non-decreasing sequence of positive real numbers satisfying $1 < p_1 \leq p_2 \leq p_3 \leq \cdots \leq p_k \leq \cdots \to \infty$.
Its set of generalized integers is the non-decreasing sequence $1 = n_0 < n_1 \leq n_2 \leq \cdots$ arising as all possible products of generalized primes and taking multiplicities into account (each of them occurs as many times as it can be represented by $p_{\nu_1}^{\alpha_1} \cdots p_{\nu_m}^{\alpha_m}$, $\nu_j < \nu_{j+1}$).

Definition:
$$Li(x) = \int_1^x \frac{1-\frac{1}{u}}{\log u}\, du$$

Define $\pi(x) = \sum_{p_k \leq x} 1$ and $N(x) = \sum_{n_k \leq x} 1$

Central problem: Study when
$$\pi(x) = \int_1^x \frac{1-\frac{1}{u}}{\log u}\, du + E_1(x) \quad \left(E_1(x) = o\left(\frac{x}{\log x}\right)\right)$$
and
$$N(x) = \rho x + E_2(x), \quad (\rho > 0;\ E_2(x) = o(x))$$
and the relationship between E_1 and E_2 in these asymptotic estimates.

1.1 Weakest error terms: The following theorems give the weakest error terms under sharp conditions (at least when sharpness is interpreted within the family of asymptotic estimates considered in their statements)

Theorem 1 (Beurling, 1937) If $(\rho > 0)$

$$(N_\beta) \quad N(x) = \rho x + O\left(\frac{x}{\log^\beta x}\right),$$

for some $\beta > \frac{3}{2}$, then

$$(1.1) \quad \pi(x) = Li(x) + o\left(\frac{x}{\log x}\right).$$

Moreover, there is a number system for which $(N_{\frac{3}{2}})$ holds but for which (1.1) fails.

Theorem 2 (Diamond, 1977). If

$$(P_\alpha) \quad \pi(x) = Li(x) + O\left(\frac{x}{\log^\alpha x}\right)$$

for some $\alpha > 1$, then

$$(1.2) \quad N(x) = \rho x + o(x) \quad (\text{for some } \rho > 0).$$

However, there is a number system for which (P_1) holds (or even $\pi(x) = Li(x) + o\left(\frac{x}{\log x}\right)$) but (1.2) fails.

Improvements to these results have been given by many authors (e.g., Kahane (1997), Schlage-Puchta–Vindas (2012), Zhang (2015), Debruyne–Vindas (2019)).

$\boxed{2}$ Quantitative versions.

Diamond showed the following quantitative version of Theorem 2.

Theorem 3 (Diamond, 1970) If $\alpha > 3$, then
$(P_\alpha) \Rightarrow (N_{\alpha-3})$.

Problem 1: Let $\beta^*(\alpha)$ be the supremum of all β admissible in the implication $(P_\alpha) \Rightarrow (N_\beta)$. Find β^* ///

Remark 1: One might also show that for (large enough) β, there is α s.t. $(N_\beta) \Rightarrow (P_\alpha)$ and define the analog best exponent $\alpha^*(\beta)$ and study the corresponding problem ///

Problem 1 is apparently hard. Theorem 3 gives the lower bound $\alpha - 3 \leq \beta^*(\alpha)$. If we want to find an upper bound for $\beta^*(\alpha)$, we need to find a suitable example of a number system for which (P_α) holds and the oscillation of the remainder in (N_β) gets as high as we can. Simple examples show that $\beta^*(\alpha) \le \alpha$. The example we study in Section $\boxed{4}$ also shows this upper bound. If we want to give a lower bound, we have to improve Theorem 3. Here is a little improvement we got that becomes meaningful in $2 \le \alpha < 5$ (where $\frac{\alpha-1}{2} > \alpha - 3$).

Theorem 4 Let $2 < \alpha < 5$. If (P_α) holds then $(N_{\frac{\alpha-1}{2}-\varepsilon})$ is true for all ε. //

We think there could be some room for improvements up to the following exponent.

Conjecture 1. $\beta^*(\alpha) = \alpha - 1$.

$\boxed{3}$ Continuous number systems

A very much used idea in this field when looking for examples is first to construct continuous analogs of number systems and then to discretize them (in the last two decades a systematic approach to the discretization is via probability arguments, namely, random approximation schemes).

We extend here the notion of number systems. Given two (Borel) measures dA and dB on $[1,\infty)$, their (Mellin) convolution is defined as the measure $dA * dB$ having distribution function

$$\int_{1^-}^{x} dA*dB = \iint_{1\le t\cdot u\le x} dA(u)\,dB(t).$$

The exponential of dA is $\exp(dA) = \delta_1 + dA + \frac{dA*dA}{2} + \cdots$ here δ_1 is the Dirac delta concentrated at 1.

Finally, a generalized number system is a pair (Π, N) of right continuous non-decreasing functions on $[1,\infty)$ satisfying $\Pi(0)=0$, $N(0)=1$, and linked via the relation

$$dN = \exp(d\Pi).$$

The Mellin transform version of it is

$$\zeta(s) := \int_{1^-}^{\infty} x^{-s}\,dN(x) = \exp\left(\int_{1}^{\infty} x^{-s}\,d\Pi(x)\right)$$

For a discrete number system $\Pi(x) = \sum_{\alpha_k \le x} \frac{1}{k\,\alpha^k}$, and the above zeta-function relation takes the form of the Euler product formula for the Beurling zeta function:

$$\zeta(s) = \sum_k \frac{1}{n_k^s} = \prod_{p_k} \frac{1}{1-\frac{1}{p_k^s}}.$$

4 A continuous number system

We study the continuous number system with Π:

$$\Pi(x) = \int_1^x \frac{1-\cos(\log^\alpha u)}{\log u}\,du, \quad \text{where } \alpha>1.$$

At the beginning I thought it would deliver an upper bound for the best exponent $\beta^*(\alpha)$ defined in Section 2; however, it failed and only gives the trivial bound $\beta^*(\alpha) \le \alpha$. Anyways, it gives the indication that it is by means not easy to improve this trivial upper bound. First we show Π satisfies (P_α) and α is the best exponent for this Π.

Lemma 1:

$$\Pi(x) = Li(x) - \frac{x}{\log^\alpha x}\left(\frac{\sin x^\alpha}{\alpha} + \frac{\cos x^\alpha}{\alpha^2}\right) + O\left(\frac{x}{\log^{\alpha+1} x}\right)$$

$$= Li(x) + O\left(\frac{x}{\log^\alpha x}\right) = Li(x) + \Omega_\pm\left(\frac{x}{\log^{\alpha+1} x}\right) \tag{6}$$

Proof. This can be shown integrating by parts and slightly redoing the proof of [1, Proposition 3.2].

The zeta function can be expressed as $(s=\sigma+it)$

$$\zeta(s)=\exp\left(\int_1^\infty \frac{(1-\cos(\log^\alpha u))}{\log u}\frac{du}{u^s}\right),$$

a function that we have studied in the past in [1,2]. For instances, $(s-1)\zeta(s)$ is an entire function [Sect. 3, 2] and $\zeta(s)$ has a simple pole with residue [Theorem 3.1, 1]

$$\rho_\alpha := \boxed{\operatorname{Res}_{s=1}\zeta(s)=\exp\left(-\gamma\left(1-\frac{1}{\alpha}\right)\right)}$$

where γ is the Euler-Mascheroni constant.

Our goal in this section is to obtain a big-O estimate for N corresponding to this Π. Actually, studying the asymptotic behavior of ζ and its derivatives on $\operatorname{Re} s=1$ and using (a variant of) a Tauberian theorem of G. Debruyne and myself, one can deduce that (with ρ_α as above):

$$N(x)=\rho_\alpha x+O\left(\frac{x}{\log x}\right).$$

Slightly improving my argument below, I think one can get $N(x)=\rho_\alpha x+\Omega\left(\frac{x}{\log^\alpha x}\right)$, but I'll skip that.

We can write

$$\log \zeta(s) = -\log(s-1) - \gamma - K(s),$$

with

$$\boxed{K(s) = \mathrm{F.p.}\int_0^\infty \frac{e^{-(s-1)u}\cos u^\alpha}{u}\,du}$$

$$:= \int_0^1 \frac{e^{-(s-1)u}\cos u^\alpha - 1}{u} + \int_1^\infty \frac{e^{-(s-1)u}\cos u^\alpha}{u}\,du, \quad \mathrm{Re}\, s > 1.$$

<u>Lemma 2.</u> The function K is entire of finite (growth) order $\frac{\alpha}{\alpha-1}$.

<u>Proof.</u> We use the analytic continuation trick from [2, Sect. 3]. If we set $s-1=iz$ and

$$(4.1) \qquad F(z) = \mathrm{F.p.}\int_0^\infty \exp(iu^\alpha - izu)\,du,$$

we can write

$$K(s) = \frac{1}{2}\left(F(z) + \overline{F(-\bar{z})}\right).$$

So, it is enough to show that F satisfies the stated property. Changing the integration contour in (4.1) to the line $\arg u = \frac{\pi}{2\alpha}$, we get

$$F(z) = \frac{\pi}{2\alpha} i + \int_0^1 \frac{\exp(-u^\alpha - i e^{\frac{\pi}{2\alpha}} z \cdot u) - 1}{u}\,du$$

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$$+\int_1^\infty \exp(-u^\alpha - i e^{\frac{\pi}{2\alpha}} \cdot z \cdot u)\,\frac{du}{u}.$$

It is very easy to see that the first integral is $\ll |z| e^{|z|} \ll_\varepsilon e^{\varepsilon |z|^{\frac{\alpha}{\alpha-1}}}$. The second integral is

$$\ll \left(\int_1^\infty \frac{\exp(-\varepsilon u^\alpha)}{u}\,du\right) \cdot \max_{u \geq 1} \exp\left(-(1-\varepsilon)u^\alpha + |z|u\right)$$

$$\ll_\varepsilon \exp\left((1+\varepsilon)|z|^{\frac{\alpha}{\alpha-1}}\left(\alpha^{-\frac{1}{\alpha-1}}\left(1-\frac{1}{\alpha}\right)\right)\right).$$

We now compute the behavior of f on the half-plane $\operatorname{Re} s \geq 1$.

<u>Lemma 3</u>: The continuous function $f(s) - \frac{\rho_\alpha}{s-1}$ tends to 1 as $|s| \to \infty$ in the closed half-plane $\operatorname{Re} s \geq 1$.

<u>Proof</u>. By [1, Theorem 3.1 (see Eq. (3.4))], we have that $f(s) - \frac{\rho_\alpha}{s-1} \longrightarrow 1$ on $\operatorname{Re} s = 1$. Since it is the Laplace transform, it belongs to $H^\infty(\{s : \operatorname{Re} s \geq 1\})$, the Hardy space on the right half-plane.

The Phragmen-Lindelöf principle (or a Poisson integral argument) yields the result.

We now get asymptotics for $\log f(s)$ in a certain region.

Lemma 4: We have that

$$(4.2)\quad \log \mathcal{S}(s) + \frac{C}{(s-1)^{\alpha}} \sim A\left(\frac{i}{s-1}\right)^{\frac{\alpha}{2(\alpha-1)}} \exp\left(-iB\left(\frac{s-1}{i}\right)^{\frac{\alpha}{\alpha-1}}\right)$$

uniformly on the region

$$(4.3)\quad 1 - \frac{1}{|t|^{\frac{1}{\alpha-1}}} \leq \sigma \leq 1, \ |t| \geq 1.$$

where $C = \Gamma(\alpha)\exp(i\pi(1-\alpha/2))$,

$$A = e^{i\frac{\pi}{4}}\sqrt{\frac{2\pi}{\alpha^{\frac{3}{\alpha-1}}(\alpha-1)}} \quad \text{and} \quad B = \alpha^{-\frac{1}{\alpha-1}}\left(1-\frac{1}{\alpha}\right).$$

Moreover, $\mathcal{S}(s)$ is bounded in this region.

Proof. Since $\mathcal{S}(\bar{s}) = \mathcal{S}(s)$, we may assume $1 \leq t$. We have shown in [2, Proposition 1, Eq. (4.1), and (4.3)] that (4.1) holds on any ray of the sector $0 \leq \arg\left(\frac{s-1}{i}\right) < \frac{\pi}{\alpha}$.

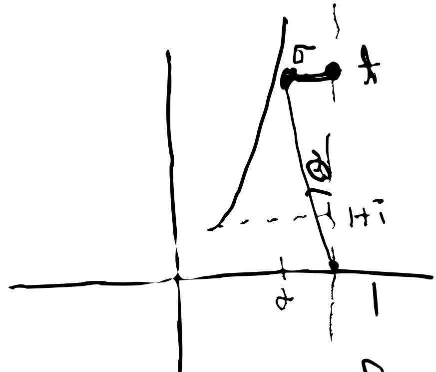
By Lemma 2 and the Phragmén-Lindelöf principle on sectors, the asymptotics (4.1) hold uniformly on any sector $0 \leq \arg\left(\frac{s-1}{i}\right) \leq \theta_0$ if $\theta_0 < \frac{\pi(\alpha-1)}{\alpha}$, in particular on (4.2).

It is enough to show that term inside the exponential of (4.1) has bounded real part. If we write points there as $s = \sigma + it = 1 + iRe^{i\theta}$, the real part of this term is (note $R\theta \to \infty$)

as $|s|\to\infty$) :

$$0 \leq \mathrm{Re}\left(-iB(Re^{i\theta})^{\frac{\alpha}{\alpha-1}}\right) = B\sin\left(\frac{\alpha\theta}{\alpha-1}\right)R^{\frac{\alpha}{\alpha-1}}$$

$$\sim \alpha^{-\frac{1}{\alpha-1}} R\sin\theta\, R^{\frac{1}{\alpha-1}} \sim \alpha^{\frac{-1}{\alpha-1}}(1-\sigma)t^{\frac{1}{\alpha-1}} \leq \alpha^{\frac{-1}{\alpha-1}}.$$



$R \sim t$

$R\sin\theta = 1-\sigma \leq \frac{1}{t^{\alpha-1}}$

Using a Tauberian theorem of Batty and Duyckaerts, (see Theorem 1 in [2]), Lemma 3 and 4 yield the following corollary:

__Corollary 1:__ $N(x) = \rho_\alpha x + O\left(x\left(\frac{\log_2 x}{\log x}\right)^{\alpha-1}\right), x\to\infty$

We will use the asymptotic formula (4.1) to improve Corollary 1 when $\alpha > \frac{3}{2}$.

__Theorem 4:__ Let $\alpha > \frac{3}{2}$. Then

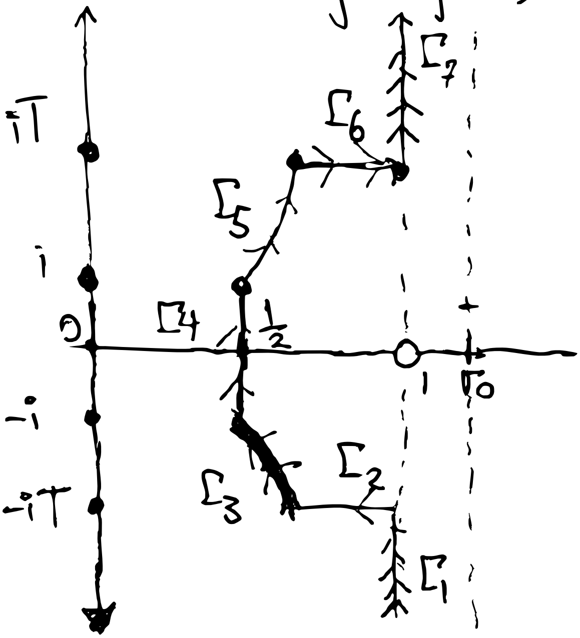
$$N(x) = \rho_\alpha x - \frac{x}{\log^\alpha x}\left(\frac{\sin x^\alpha}{\alpha} + \frac{\cos x^\alpha}{\alpha^2}\right) + O_\eta\left(\frac{x}{\log^{\alpha+\eta} x}\right)$$

where $\eta = 1$ if $\alpha \geq 2$ and $0 < \eta < \frac{\alpha^3}{3\alpha-2}$ if $\frac{3}{2} < \alpha \leq 2$.

Proof. We use Perron's inversion formula:

$$N(x) = \frac{1}{2\pi i}\,\mathrm{p.v.}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{\zeta(s)}{s}\,x^{s}\,ds\,, \quad \sigma_0>1.$$

We choose $\sigma_0 = 1+\frac{1}{\log x}$, so that x^{s-1} remains bounded as $\sigma \leq \sigma_0$. We switch the integration contour as in the following figure, where T is a parameter to choose.



$\Gamma_7 \cup \Gamma_1$: σ_0+it: $|t| \geq T$

$\Gamma_6 \cup \Gamma_2$: $\sigma \pm iT$: $1-\frac{1}{2T^{\frac{1}{\alpha-1}}} \leq \sigma \leq \sigma_0$

$\Gamma_5 \cup \Gamma_3$: $1-\frac{1}{2|t|^{\frac{1}{\alpha-1}}}+it$: $1 \leq |t| \leq T$

Γ_4: $\frac{1}{2}+it$: $|t| \leq 1$

We set $\Gamma = \bigcup_{i=1}^{7} \Gamma_i$ and since we have passed through the pole, we have

$$N(x) = \rho_\alpha x + \frac{1}{2\pi i}\,\mathrm{p.v.}\int_{\Gamma} \frac{\zeta(s)}{s}\,x^{s}\,ds.$$

Now on $\Gamma \setminus \Gamma_4$, we have $\log \zeta(s) = O\left(\frac{1}{t^{\frac{\alpha}{2(\alpha-1)}}}\right)$

by Lemma 4 (notice that $\frac{\alpha}{2(\alpha-1)} \leq \alpha$ when $\alpha \geq \frac{3}{2}$).

Since every involved integral is $O(x^{\frac{1}{2}})$ on Γ_4 and on $\Gamma \setminus \Gamma_4$, $\zeta(s) = \exp(\log \zeta(s)) = 1 + \log\zeta(s) + O((\log\zeta(s))^2)$

$$= 1 + \log\zeta(s) + A\left(\frac{i}{s-1}\right)^{\frac{\alpha}{\alpha-1}} \exp\left(-2iB\left(\frac{s-1}{i}\right)^{\frac{\alpha}{\alpha-1}}\right) + O\left(\frac{1}{|t|^{\frac{3\alpha}{\alpha-1}}}\right),$$

again by Lemma 4. So,

$$N(x) = \rho_\alpha x + \frac{\text{p.v.}}{2\pi i}\int_\Gamma \frac{x^s}{s}ds + \frac{\text{p.v.}}{2\pi i}\int_\Gamma \frac{x^s \log\zeta(s)}{s}ds + \int_{\Gamma_2\cup\Gamma_3\cup\Gamma_5\cup\Gamma_6} x^s O\left(\frac{1}{t^{1+\frac{3\alpha}{2(\alpha-1)}}}\right)ds$$

$$+ \frac{A}{2\pi i}\int_{\Gamma_1\cup\Gamma_7} \frac{x^s}{s} A\left(\frac{i}{s-1}\right)^{\frac{\alpha}{\alpha-1}} \exp\left(-2iB\left(\frac{s-1}{i}\right)^{\frac{\alpha}{\alpha-1}}\right)ds + O(x^{\frac{1}{2}})$$

$$=: \rho_\alpha x + I_1(x) + I_2(x) + I_3(x) + I_4(x)\ O(x^{\frac{1}{2}})$$

The integral I_1 is easy to handle; in fact it is the Mellin-transform of the Dirac delta δ_1 concentrated at 1, so that $1 = \int_1^x \delta_1 = \frac{1}{2\pi i}\text{p.v.}\int_\Gamma \frac{x^s}{s}ds$, $x > 1$, by Perron inversion formula. So $I_1(x)$ gets absorbed into the error term. The integrals defining I_3 on $\Gamma_2\cup\Gamma_6$ are $O\left(\frac{1}{T^{1+\frac{3\alpha}{2(\alpha-1)}}}\right)$. On the other hand,

$$\int_{\Gamma_3\cup\Gamma_5} x^s O\left(\frac{1}{t^{1+\frac{3\alpha}{\alpha-1}}}\right)ds \ll x\int^T \exp\left(-\frac{1}{2}\left(\frac{\log^{\alpha-1} x}{t}\right)^{\frac{1}{\alpha-1}}\right)\frac{dt}{t^{1+\frac{3\alpha}{2(\alpha-1)}}}$$

$$\ll x\exp\left(-\frac{1}{2}\left(\frac{\log^{\alpha-1} x}{T}\right)^{\frac{1}{\alpha-1}}\right), \text{ so that}$$

$$I_3(x) \ll \frac{1}{T^{1+\frac{3\alpha}{2(\alpha-1)}}} + \exp\left(-\frac{1}{2}\left(\frac{\log^{\alpha-1}}{T}\right)^{\frac{1}{\alpha-1}}\right).$$

We can estimate the integral $I_4(X)$ by using van der Corput's inequality. It is enough to consider Γ_7. We write $f(t) = t\log x - 2B\,t^{\frac{\alpha}{\alpha-1}}$. Up to a constant factor the integral we are interested in estimating is

$$X\int_T^{\infty} \frac{\exp(if(t))}{(1+it)\,t^{\frac{\alpha}{\alpha-1}}}\,dt.$$

We note that $|f''(t)| \gg t^{\frac{\alpha}{\alpha-1}-2} \gg Y^{\frac{\alpha}{\alpha-1}-2}$ on any interval $[T, Y]$. A classical lemma of van der Corput (cf. [3, Theorem I.6.3]) gives

$$\int_T^Y \exp(if(t))\,dt \ll Y^{1-\frac{\alpha}{2(\alpha-1)}}$$

Integrating by parts for the range $[T, Y]$ and estimating trivially on $[Y, \infty)$, we get

$$I_4(x) \ll_x \frac{Y^{1-\frac{\alpha}{2(\alpha-1)}}}{T^{1+\frac{\alpha}{\alpha-1}}} + \frac{X}{Y^{\frac{\alpha}{\alpha-1}}} \ll \frac{X}{T^{\frac{\alpha(4\alpha-2)}{(\alpha-1)(3\alpha-2)}}}$$

uppon choosing $Y = T^{\frac{4\alpha-2}{3\alpha-2}}$. This exponent of T is worse than $1+\frac{3\alpha}{2(\alpha-1)}$. Summarizing,

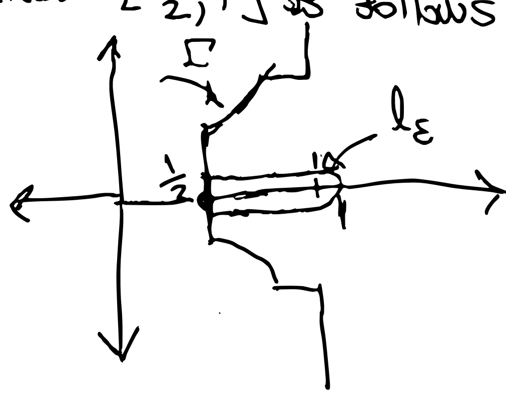
$$I_1(x)+I_3(x)+I_4(x) \ll x\exp\left(-\frac{1}{2}\left(\frac{\log^{\alpha-1}x}{T}\right)^{\frac{1}{\alpha-1}}\right)$$
$$+\, x\,T^{-\frac{\alpha}{(\alpha-1)}\frac{(4\alpha-2)}{(3\alpha-2)}}$$

If we choose $T=\log^{\beta}x$ with $\beta=\dfrac{(\alpha+\omega)(\alpha-1)(3\alpha-2)}{\alpha(4\alpha-2)}$ with $\boxed{\dfrac{\alpha^2}{3\alpha-2}>\omega}$ we make sure that $\beta<\alpha-1$, so that $I_1(x)+I_3(x)+I_4(x) \ll \dfrac{x}{\log^{\alpha+\omega}x}$.

Up to now we have therefore seen that

$$N(x)=\rho_\alpha x+\frac{1}{2\pi i}\,\text{p.v.}\int_{\Gamma}\frac{x^s}{s}\log\zeta(s)\,ds+O\left(\frac{x}{\log^{\alpha+\omega}x}\right)$$

For the integral we return to the contour $\sigma_0=1+\frac{1}{\log x}$ where the Perron integral is $\Pi(x)$. For it we add and subtract part of a Hankel contour the interval $[\frac{1}{2},1]$ as follows:



We switch $\Gamma \cup l_\varepsilon$ back to $\sigma_0 = 1 + \frac{1}{\log x}$, take $\varepsilon \to 0$, & use that the Perron integral on σ_0 is precisely $\Pi(x)$. So,

$$N(x) = \rho_\alpha x + \Pi(x) + O\left(\frac{x}{\log^{\alpha+\omega} x}\right)$$

$$- \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\frac{1}{2}}^{1} \left(\frac{x^{i\varepsilon} \log \zeta(\sigma + i\varepsilon)}{\sigma + i\varepsilon} - \frac{x^{-i\varepsilon} \log \zeta(\sigma - i\varepsilon)}{\sigma - i\varepsilon} \right) x^{\sigma} d\sigma.$$

Since $\log \zeta(s) + \log(s-1)$ is a continuous function the limit is

$$\frac{1}{2\pi i} \int_{\frac{1}{2}}^{1} 2\pi i \exp(\sigma \log x) \frac{d\sigma}{\sigma} = Li(x) - Li(\sqrt{x}) + O(\log_2 x)$$

$$= Li(x) + O(\sqrt{x}).$$

Using Lemma 1, finally we conclude that $(\eta = \min\{1, \omega\})$

$$N(x) = \rho_\alpha x + \Pi(x) - Li(x) + O\left(\frac{x}{\log^{\alpha+\eta} x}\right)$$

$$= \rho_\alpha x - \frac{x}{\log^\alpha x}\left(\frac{\sin x^\alpha}{\alpha} + \frac{\cos^\alpha x}{\alpha^2}\right) + O\left(\frac{x}{\log^{\alpha+\eta} x}\right)$$

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