The Prime Number Theorem for Generalized Integers. New Cases

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The prime number theorem (PNT) states that

$$\pi(x) \sim \frac{x}{\log x}, \quad x \to \infty,$$

where

$$\pi(x) = \sum_{\text{prime } p, \ p < x} 1.$$

We will consider in this talk generalizations of the PNT for Beurling’s generalized integers.
Outline

1. Abstract prime number theorems
   - Landau’s PNT
   - Beurling’s problem

2. The main theorem: Extension of Beurling’s theorem

3. A Tauberian approach
   - Ikehara’s Tauberian theorem
   - A Tauberian theorem for local pseudo-function boundary behavior

4. Other related results
In 1903, Landau essentially showed the following theorem.

- Let $1 < p_1 \leq p_2, \ldots$ be a non-decreasing sequence tending to infinity.
- Arrange all possible products of the $p_j$ in a non-decreasing sequence $1 < n_1 \leq n_2, \ldots$, where every $n_k$ is repeated as many times as represented by $p_{\nu_1}^{\alpha_1} p_{\nu_2}^{\alpha_2} \ldots p_{\nu_m}^{\alpha_m}$ with $\nu_j < \nu_{j+1}$.
- Denote $N(x) = \sum_{n_k < x} 1$ and $\pi(x) = \sum_{p_k < x} 1$.

Theorem (Landau, 1903)

If $N(x) = ax + O(x^\theta)$, $x \to \infty$, where $a > 0$ and $\theta < 1$, then

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Landau’s theorem: Examples

- **Gaussian integers** \( \mathbb{Z}[i] := \{ a + bi \in \mathbb{C} : a, b \in \mathbb{Z} \} \), with Gaussian norm \( |a + ib| := a^2 + b^2 \). If we define \( \{ p_k \}_{k=1}^\infty \) as the sequence of norms of Gaussian primes, then the sequence \( \{ n_k \}_{k=1}^\infty \) corresponds to the sequence of norms of gaussian numbers such that \( |a + ib| > 1 \). One can show that

\[
N(x) = \sum_{a,b \in \mathbb{Z}, \ a^2+b^2 < x} 1 = \pi x + O(\sqrt{x})
\]

Consequently, the PNT holds for Gaussian primes.

- Landau actually showed that if the \( \{ p_k \}_{k=1}^\infty \) corresponds to the norms of the distinct primes ideal of the ring of integers in an algebraic number field, then \( \pi(x) \sim x / \log x \).
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Beurling’s problem

In 1937, Beurling raised the question: Find conditions over $N$ which ensure the validity of the PNT, i.e., $\pi(x) \sim x / \log x$.

**Theorem (Beurling, 1937)**

If

$$N(x) = ax + O \left( \frac{x}{\log \gamma x} \right),$$

where $a > 0$ and $\gamma > 3/2$, then the PNT holds.

**Theorem (Diamond, 1970)**

Beurling’s condition is sharp, namely, the PNT does not necessarily hold if $\gamma = 3/2$. 
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We were able to relax the hypothesis of Beurling’s theorem.

**Theorem (2010, extending Beurling, 1937)**

*Suppose there exist constants* \( a > 0 \) *and* \( \gamma > 3/2 \) *such that*

\[
N(x) = ax + O\left(\frac{x}{\log^\gamma x}\right) \quad (C), \quad x \to \infty,
\]

*Then the prime number theorem still holds.*

The hypothesis means that there exists some \( m \in \mathbb{N} \) such that:

\[
\int_0^x \frac{N(t) - at}{t} \left(1 - \frac{t}{x}\right)^m \, dt = O\left(\frac{x}{\log^\gamma x}\right), \quad x \to \infty.
\]
Extension of Beurling theorem

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Theorem (2010, extending Beurling, 1937)

Suppose there exist constants $a > 0$ and $\gamma > 3/2$ such that

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We say a function $f(x) = O(x^\beta / \log^\alpha x)$ \((C, m)\), $\beta > -1$, if

$$\frac{1}{x} \int_0^x f(t) \left(1 - \frac{t}{x}\right)^{m-1} \, dt = O\left(\frac{x^\beta}{\log^\alpha x}\right).$$

- The above expression is the $m$-times iterated primitive of $f$ divided by $x^m$.
- Cesàro means have been widely used in Fourier analysis, they allow a high degree of divergence, often cancelled by oscillation.

**Examples:** \((0 < \alpha < 1)\)

- $e^x \sin e^x = O(x^{-\alpha})$ \((C, 1)\).
- $\sum_{0 \leq k \leq x} (-1)^k = 1/2 + O(x^{-\alpha})$ \((C, 1)\).
Few words about Cesàro asymptotics

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however, one can show that

$$N(x) \sim ax = ax + o(x)$$
The zeta function is the analytic function (under our hypothesis)

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{\eta_k^s}, \quad \Re s > 1.$$  

For ordinary integers it reduces to the Riemann zeta function. One has an Euler product representation

$$\zeta(s) = \prod_{k=1}^{\infty} \frac{1}{1 - \left(\frac{1}{p_k}\right)^s}, \quad \Re s > 1.$$
Functions related to generalized primes

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Define the **von Mangoldt function**

\[ \Lambda(n_k) = \begin{cases} \log p_j , & \text{if } n_k = p_j^m , \\ 0 , & \text{otherwise} . \end{cases} \]

The **Chebyshev function** is

\[ \psi(x) = \sum_{p_k^m < x} \log p_k = \sum_{n_k < x} \Lambda(n_k) . \]

One can show the PNT is equivalent to \( \psi(x) \sim x \). We also have the identity

\[ \sum_{k=1}^{\infty} \frac{\Lambda(n_k)}{n_k^s} = -\frac{\zeta'(s)}{\zeta(s)} , \quad \Re s > 1 . \]
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Ikehara’s Tauberian theorem

One of quickest ways to the PNT (for ordinary primes) is via the following Tauberian theorem:

Theorem (Ikehara, 1931, extending Landau, 1908)

Let $F(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$ be convergent for $\Re e \ s > 1$. Assume additionally that $c_n \geq 0$. If there exists a constant $\beta$ such that

$$G(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s} - \frac{\beta}{s-1} = F(s) - \frac{\beta}{s-1}, \quad \Re e \ s > 1, \quad (1)$$

has a continuous extension to $\Re e \ s = 1$, then

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Consider the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $\Re s > 1$.

- $\zeta(s) - \frac{1}{s-1}$ admits an analytic continuation to a neighborhood of $\Re s = 1$
- $\zeta(1 + it), t \neq 1$, is free of zeros

It follows that

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} - \frac{1}{s-1} = -\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$$

admits a (analytic) continuous extension to $\Re s = 1$.

Consequently,

$$\sum_{1 \leq n < x} \Lambda(n) = \psi(x) \sim x$$
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- In the case of Landau’s hypothesis: $N(x) = ax + O(x^\theta)$
  
  1. The function $\zeta(s) - \frac{a}{s - 1}$ admits an analytic continuation to a neighborhood of $\Re s = 1$
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  3. So, a variant of Ikehara theorem yields, as before, the PNT

- For Beurling’s hypothesis: $N(x) = ax + O(x/\log^{\gamma} x)$
  
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Tempered distributions

- $S(\mathbb{R})$ denotes the space of rapidly decreasing test functions, i.e.,

\[ \| \phi \|_j := \sup_{x \in \mathbb{R}, k \leq j} (1 + |x|)^j \left| \phi^{(k)}(x) \right| < \infty, \text{ for each } j \in \mathbb{N}, \]

with the Fréchet space topology defined by the above seminorms.

- Fourier transform, $\hat{\phi}(t) = \int_{-\infty}^{\infty} e^{-itx} \phi(x) \, dx$, is an isomorphism.

- The space $S'(\mathbb{R})$ is its dual, the Fourier transform is defined by

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- \( S(\mathbb{R}) \) denotes the space of rapidly decreasing test functions, i.e.,

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\| \phi \|_j := \sup_{x \in \mathbb{R}, k \leq j} (1 + |x|)^j \left| \phi^{(k)}(x) \right| < \infty, \text{ for each } j \in \mathbb{N},
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with the Fréchet space topology defined by the above seminorms.

- Fourier transform, \( \hat{\phi}(t) = \int_{-\infty}^{\infty} e^{-itx} \phi(x) \, dx \), is an isomorphism.

- The space \( S'(\mathbb{R}) \) is its dual, the Fourier transform is defined by

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\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle.
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A distribution \( f \in S'(\mathbb{R}) \) is called a **pseudo-function** if \( \hat{f} \in C_0(\mathbb{R}) \).

It is called a **local** pseudofunction if for each \( \phi \in S(\mathbb{R}) \) with compact support, the distribution \( \phi f \) is a pseudo-function. \( f \) is locally a pseudo-function if and only if the following ‘Riemann-Lebesgue lemma’ holds: for each \( \phi \) with compact support

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\lim_{|h| \to \infty} \left\langle f(t), e^{-iht} \phi(t) \right\rangle = 0
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**Corollary**

*If \( f \) belongs to \( C(\mathbb{R}) \), or more generally \( L^1_{\text{loc}}(\mathbb{R}) \), then \( f \) is locally a pseudo-function.*
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Local pseudo-function boundary behavior

Let $G(s)$ be analytic on $\Re s > \alpha$. We say that $G$ has local pseudo-function boundary behavior on the line $\Re s = \alpha$ if it has distributional boundary values in such a line, namely

$$\lim_{\sigma \to \alpha^+} \int_{-\infty}^{\infty} G(\sigma + it) \phi(t) dt = \langle f, \phi \rangle, \quad \phi \in S(\mathbb{R}) \text{ with compact support},$$

and the boundary distribution $f \in S'(\mathbb{R})$ is locally a pseudo-function.
A Tauberian theorem for local pseudo-function boundary behavior

Theorem

Let \( \{\lambda_k\}_{k=1}^{\infty} \) be such that \( 0 < \lambda_k \nearrow \infty \).

Assume \( \{c_k\}_{k=1}^{\infty} \) satisfies: \( c_k \geq 0 \) and \( \sum_{\lambda_k < x} c_k = O(x) \).

If there exists \( \beta \) such that

\[
G(s) = \sum_{k=1}^{\infty} \frac{c_k}{\lambda_k^s} - \frac{\beta}{s - 1}, \quad \Re s > 1,
\]

has local pseudo-function boundary behavior on \( \Re s = 1 \), then

\[
\sum_{\lambda_k < x} c_k \sim \beta x, \quad x \to \infty.
\]
Under $N(x) = ax + O(x/\log^\gamma x)$ (C)

Using ‘generalized distributional asymptotics’, we translated the Cesàro estimate into:

- For $\gamma > 1$, $\zeta(s) - \frac{a}{s-1}$ has continuous extension to $\Re s = 1$.
- For $\gamma > 3/2$
  - $(s-1)\zeta(s)$ is free of zeros on $\Re s = 1$.
  - $-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$ has local pseudo-function boundary behavior on the line $\Re s = 1$.
  - A Chebyshev upper estimate: $\sum_{n_k < x} \Lambda(n) = \psi(x) = O(x)$
  - So, the last Tauberian theorem implies the PNT ($\gamma > 3/2$)
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Other related results ($\gamma > 3/2$)

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*Our theorem is a proper extension of Beurling's PNT, namely, there is a set of generalized numbers satisfying the Cesàro estimate but not Beurling's one.*

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*Let $\mu$ be the Möbius function. Then,*

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