The Prime Number Theorem for Generalized Integers. New Cases

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The prime number theorem

The prime number theorem (PNT) states that

$$\pi(x) \sim \frac{x}{\log x}, \quad x \to \infty ,$$

where

$$\pi(x) = \sum_{p \text{ prime},\ p<x} 1 .$$

We will consider in this talk generalizations of the PNT for Beurling's generalized integers

Outline

1. Abstract prime number theorems
   - Landau's PNT
   - Beurling's problem
2. The main theorem: Extension of Beurling's theorem
3. A Tauberian approach
   - Ikehara's Tauberian theorem
   - A Tauberian theorem for local pseudo-function boundary behavior
4. Other related results

Landau's theorem

In 1903, Landau essentially showed the following theorem.

- Let $1 < p_1 \leq p_2, \dots$ be a non-decreasing sequence tending to infinity.
- Arrange all possible products of the p_j in a non-decreasing sequence $1 < n_1 \leq n_2, \dots,$ where every n_k is repeated as many times as represented by $p_{\nu_1}^{\alpha_1} p_{\nu_2}^{\alpha_2} \dots p_{\nu_m}^{\alpha_m}$ with $\nu_j < \nu_{j+1}$.
- Denote $N(x) = \sum_{n_k < x} 1$ and $\pi(x) = \sum_{p_k < x} 1$.

Theorem (Landau, 1903)

If $N(x) = ax + O(x^{\theta})\,, \quad x \to \infty\,,$ *where* $a > 0$ *and* $\theta < 1$, *then*

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Landau's theorem: Examples

- Gaussian integers $\mathbb{Z}[i] := \{a + b\,i \in \mathbb{C} : a, b \in \mathbb{Z}\}$, with Gaussian norm $|a + ib| := a^2 + b^2$. If we define $\{p_k\}_{k=1}^{\infty}$ as the sequence of norms of Gaussian primes, then the sequence $\{n_k\}_{k=1}^{\infty}$ corresponds to the sequence of norms of gaussian numbers such that $|a + ib| > 1$. One can show that

$$N(x) = \sum_{a,b \in \mathbb{Z},\ a^2+b^2<x} 1 = \pi x + O(\sqrt{x})$$

Consequently, the PNT holds for Gaussian primes.

- Laudau actually showed that if the $\{p_k\}_{k=1}^{\infty}$ corresponds to the norms of the distinct primes ideal of the ring of integers in an algebraic number field, then $\pi(x) \sim x/\log x$.

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Beurling's problem

In 1937, Beurling raised the question: Find conditions over N which ensure the validity of the PNT, i.e., $\pi(x) \sim x/\log x$.

Theorem (Beurling, 1937)

if

$$N(x) = ax + O\left(\frac{x}{\log^{\gamma} x}\right),$$

where $a > 0$ and $\gamma > 3/2$, then the PNT holds.

Theorem (Diamond, 1970)

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Extension of Beurling theorem

We were able to relax the hypothesis of Beurling's theorem.

Theorem (2010, exdending Beurling, 1937)

Suppose there exist constants $a > 0$ and $\gamma > 3/2$ such that

$$N(x) = ax + O\left(\frac{x}{\log^{\gamma} x}\right) \quad (\mathrm{C}) \, , \quad x \to \infty \, ,$$

Then the prime number theorem still holds.

The hypothesis means that there exists some $m \in \mathbb{N}$ such that:

$$\int_{0}^{x} \frac{N(t) - at}{t} \left(1 - \frac{t}{x}\right)^{m} \mathrm{d}t = O\left(\frac{x}{\log^{\gamma} x}\right) \, , \quad x \to \infty \, .$$

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Few words about Cesàro asymptotics

We say a function $f(x) = O(x^{\beta}/\log^{\alpha} x) \quad (\mathrm{C}, m)$, $\beta > -1$, if

$$\frac{1}{x}\int_{0}^{x} f(t)\left(1-\frac{t}{x}\right)^{m-1} \mathrm{d}t = O\left(\frac{x^{\beta}}{\log^{\alpha} x}\right) .$$

- The above expression is the m-times iterated primitive of f divided by x^{m}
- Cesàro means have been widely used in Fourier analysis, they allow a high degree of divergence, often cancelled by oscillation.

Examples: $(0 < \alpha < 1)$

- $e^{x} \sin e^{x} = O(x^{-\alpha}) \quad (\mathrm{C}, 1)$.
- $\sum_{0 \leq k \leq x} (-1)^{k} = 1/2 + O(x^{-\alpha}) \quad (\mathrm{C}, 1)$.

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however, one can show that

$$N(x) \sim ax = ax + o(x)$$

Functions related to generalized primes

The zeta function is the analytic function (under our hypothesis)

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{n_k^s}\ , \quad \Re e\, s > 1\ .$$

For ordinary integers it reduces to the Riemann zeta function.
One has an Euler product representation

$$\zeta(s) = \prod_{k=1}^{\infty} \frac{1}{1 - \left(\frac{1}{p_k}\right)^s}\ , \quad \Re e\, s > 1\ .$$

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Define the von Mangoldt function

$$\Lambda(n_k) = \begin{cases} \log p_j\,, & \text{if } n_k = p_j^m\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

The Chebyshev function is

$$\psi(x) = \sum_{p_k^m < x} \log p_k = \sum_{n_k < x} \Lambda(n_k)\,.$$

On can show the PNT is equivalent to $\psi(x) \sim x$. We also have the identity

$$\sum_{k=1}^{\infty} \frac{\Lambda(n_k)}{n_k^s} = -\frac{\zeta'(s)}{\zeta(s)}\,, \quad \Re e\, s > 1\,.$$

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One of quickest ways to the PNT (for ordinary primes) is via the following Tauberian theorem:

Theorem (Ikehara, 1931, extending Landau, 1908)

Let $F(s)=\sum_{n=1}^{\infty} c_n/n^s$ be convergent for $\Re e\, s>1$. Assume additionally that $c_n\geq 0$. If there exists a constant β such that

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The PNT (for ordinary prime numbers)

Consider the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $\Re e\, s > 1$.

- $\zeta(s) - \frac{1}{s-1}$ admits an analytic continuation to a neighborhood of $\Re e\, s = 1$
- $\zeta(1+it)$, $t \neq 1$, is free of zeros

It follows that

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} - \frac{1}{s-1} = -\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$$

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Comments on Landau and Beurling PNTs

- In the case of Landau's hypothesis: $N(x) = ax + O(x^{\theta})$
  - (1) The function $\zeta(s) - \frac{a}{s-1}$ admits an analytic continuation to a neighborhood of $\Re e\, s = 1$
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  - (3) So, a variant of Ikehara theorem yields, as before, the PNT
- For Beurling's hypothesis: $N(x) = ax + O(x/\log^{\gamma} x)$
  - (1') If $\gamma > 2$, the function $\zeta(s) - \frac{a}{s-1}$ admits a continuously differentiable extension to $\Re e\, s = 1$ (not true for $3/2 < \gamma \leq 2$)
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Tempered distributions

- $\mathcal{S}(\mathbb{R})$ denotes the space of rapidly decresiang test functions, i.e.,

$$\|\phi\|_j := \sup_{x\in\mathbb{R},k\leq j} (1+|x|)^j \left|\phi^{(k)}(x)\right| < \infty \ , \ \text{for each } j \in \mathbb{N} \ ,$$

  with the Fréchet space topology defined by the above seminorms.
- Fourier transform, $\hat{\phi}(t) = \int_{-\infty}^{\infty} e^{-itx}\phi(x)\,\mathrm{d}x$, is an isomorphism.
- The space $\mathcal{S}'(\mathbb{R})$ is its dual,the Fourier transform is defined by

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- Fourier transform, $\hat{\phi}(t) = \int_{-\infty}^{\infty} e^{-itx}\phi(x)\,\mathrm{d}x$, is an isomorphism.
- The space $\mathcal{S}'(\mathbb{R})$ is its dual, the Fourier transform is defined by

$$\left\langle \hat{f}, \phi \right\rangle = \left\langle f, \hat{\phi} \right\rangle\ .$$

Tempered distributions

- $\mathcal{S}(\mathbb{R})$ denotes the space of rapidly decresiang test functions, i.e.,

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Pseudo-functions

A distribution $f \in \mathcal{S}'(\mathbb{R})$ is called a pseudo-function if $\hat{f} \in C_0(\mathbb{R})$.

It is called a local pseudofunction if for each $\phi \in \mathcal{S}(\mathbb{R})$ with compact support, the distribution ϕf is a pseudo-function.

f is locally a pseudo-function if and only if the following 'Riemann-Lebesgue lemma' holds: for each ϕ with compact support

$$\lim_{|h|\to\infty} \left\langle f(t), e^{-iht}\phi(t) \right\rangle = 0$$

Corollary

If f belongs to $C(\mathbb{R})$, or more generally $L^1_{\mathrm{loc}}(\mathbb{R})$, then f is locally a pseudo-function.

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Local pseudo-function boundary behavior

Let $G(s)$ be analytic on $\Re e\, s > \alpha$. We say that G has local pseudo-function boundary behavior on the line $\Re e\, s = \alpha$ if it has distributional boundary values in such a line, namely

$$\lim_{\sigma \to \alpha^+} \int_{-\infty}^{\infty} G(\sigma + it)\phi(t)\mathrm{d}t = \langle f, \phi \rangle \ , \quad \phi \in \mathcal{S}(\mathbb{R}) \text{ with compact support} \ ,$$

and the boundary distribution $f \in \mathcal{S}'(\mathbb{R})$ is locally a pseudo-function.

A Tauberian theorem
for local pseudo-function boundary behavior

Theorem

Let $\{\lambda_k\}_{k=1}^{\infty}$ *be such that* $0 < \lambda_k \nearrow \infty$.
Assume $\{c_k\}_{k=1}^{\infty}$ *satisfies:* $c_k \geq 0$ *and* $\sum_{\lambda_k < x} c_k = O(x)$.
If there exists β *such that*

$$G(s) = \sum_{k=1}^{\infty} \frac{c_k}{\lambda_k^s} - \frac{\beta}{s-1}\, , \quad \Re e\, s > 1\, , \tag{3}$$

has local pseudo-function boundary behavior on $\Re e\, s = 1$*, then*

$$\sum_{\lambda_k < x} c_k \sim \beta x\, , \quad x \to \infty\, . \tag{4}$$

Under $N(x) = ax + O(x/\log^{\gamma} x)$ (C)

Using 'generalized distributional asymptotics', we translated the Cesàro estimate into:

- For $\gamma > 1$, $\zeta(s) - \dfrac{a}{s-1}$ has continuous extension to $\Re e\, s = 1$.
- For $\gamma > 3/2$
  - $(s-1)\zeta(s)$ is free of zeros on $\Re e\, s = 1$.
  - $-\dfrac{\zeta'(s)}{\zeta(s)} - \dfrac{1}{s-1}$ has local pseudo-function boundary behavior on the line $\Re e\, s = 1$
  - A Chebyshev upper estimate: $\sum_{n_k<x} \Lambda(n) = \psi(x) = O(x)$
  - So, the last Tauberian theorem implies the PNT $(\gamma > 3/2)$

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Other related results ($\gamma > 3/2$)

Theorem

Our theorem is a proper extension of Beurling's PNT, namely, there is a set of generalized numbers satisfying the Cesàro estimate but not Beurling's one.

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Let μ be the Möbius function. Then,

$$\sum_{k=1}^{\infty} \frac{\mu(n_k)}{n_k} = 0 \text{ and } \lim_{x\to\infty} \frac{1}{x} \sum_{n_k<x} \mu(n_k) = 0 \, .$$

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