# Weyl asymptotic formulas for infinite order ΨDOs and Sobolev type spaces. Part II.

Jasson Vindas
jasson.vindas@UGent.be

(collaborative work with Stevan Pilipović and Bojan Prangoski)

Department of Mathematics
Ghent University

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# Introduction

The Weyl asymptotic formula relates the spectral asymptotics of a $\Psi$DO with properties of its symbol.

Let $a(x, D) = \sum_{|\alpha|+|\beta|\leq m} c_{\alpha,\beta} x^{\beta} D^{\alpha}$ be a positive (globally) elliptic Shubin PDO.

Its spectrum consists of a sequence of eigenvalues $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \dots$, whose counting function

$$N(\lambda) = \sum_{\lambda_j \leq \lambda} 1$$

behaves according the Weyl law:

**Weyl asymptotic formula**

$$N(\lambda) \sim \frac{1}{(2\pi)^d} \iint_{a(x,\xi)<\lambda} dx d\xi, \quad \lambda \to \infty.$$

Goal: Spectral asymptotics for infinite order $\Psi DOs$.

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# Symbol classes

Let $M_p$ and $A_p$ be weight sequences such that

- $M_p$ satisfies $(M.1)$, $(M.2)$, and $(M.3)$.
- $A_p$ satisfies $(M.1)$, $(M.2)$, $(M.3)'$, and $(M.4)$.
- $A_p \subset M_p$.
- Let $0 < \rho \leq 1$ such that $A_p \subset M_p^{\rho}$.

Associated function: $M(t) = \sup_{p\in\mathbb{N}} \ln_+ \dfrac{t^p}{M_p}, \quad t \in [0,\infty)$.

Define $\Gamma^{M_p,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h, m)$ as the space of all $a \in C^\infty(\mathbb{R}^{2d})$ such that

$$\sup_{\alpha\in\mathbb{N}^{2d}} \sup_{w\in\mathbb{R}^{2d}} \frac{|D^\alpha a(w)| \langle w\rangle^{\rho|\alpha|} e^{-M(m|w|)}}{h^{|\alpha|} A_{|\alpha|}} < \infty.$$

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# Symbol classes and infinite order ΨDOs

## Symbol Classes $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$

$$\Gamma^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d}) = \varinjlim_{m\to\infty} \varprojlim_{h\to 0} \Gamma^{M_p,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h, m)$$

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$\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ common notation for $* = (M_p), \{M_p\}$.

Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$.

- Its $\tau$-quantization $\operatorname{Op}_\tau(a) : \mathcal{S}^*(\mathbb{R}^d) \to \mathcal{S}^*(\mathbb{R}^d)$ is continuous.
- We write $a^w = \operatorname{Op}_{1/2}(a)$ for its Weyl quantization.
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# Spectral asymptotics

- Consider a real-valued hypoelliptic $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ with $a(w) \to \infty$ as $|w| \to \infty$.
- Denote still by $a^w$ the closure of the unbounded self-adjoint operator on $L^2(\mathbb{R}^d)$ induced by its Weyl quantization.
- As explained in the talk by Prangoski, the spectrum of $a^w$ is given by an unbounded sequence of eigenvalues (multiplicities taken into account)

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \ldots$$

**Problem: Spectral asymptotics**

Denote the spectral counting function of the operator $a^w$ as

$$N(\lambda) = \sum_{\lambda_j \leq \lambda} 1 = \#\{j \in \mathbb{N} \mid \lambda_j \leq \lambda\}.$$

Goal: Asymptotic behavior of $N$ under mild assumptions on $a$.

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# The setup: Tauberian problem

Assume additionally that $a$ satisfies $a(w)/\ln|w| \to \infty$.

Analysis of the associated heat semigroup yields the heat asymptotics ($t \to 0^+$)

$$\int_0^\infty e^{-t\lambda} dN(\lambda) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{-ta(w)} dw + O\left( \int_{\mathbb{R}^{2d}} \frac{e^{-ta(w)/4}}{\langle w \rangle^{2\rho}} dw \right).$$

The problem is now of Tauberian character: find conditions on the symbol $a$ to 'unaverage' this and translate it into asymptotics for $N(\lambda)$.

We use growth comparison functions $f : [0, \infty) \to \mathbb{R}_+$ such that

- eventually increasing, absolutely continuous

Set $\sigma(\lambda) = (f^{-1}(\lambda))^{2d}$ for large $\lambda$.

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# Weyl formula: infinite order case

For operators that are of infinite order, we have:

**Theorem**

*Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ hypoelliptic, let $f$ satisfy*

$$\lim_{y\to\infty} \frac{yf'(y)}{f(y)} = \infty,$$

*and let $\Phi$ be a positive continuous function on the sphere $\mathbb{S}^{2d-1}$. Suppose that for each $\varepsilon \in (0,1)$ there are positive constants $c_\epsilon, C_\epsilon, B_\epsilon > 0$ such that*

$$c_\varepsilon f((1-\varepsilon)r\Phi(\vartheta)) \leq a(r\vartheta) \leq C_\varepsilon f((1+\varepsilon)r\Phi(\vartheta)),$$

*for all $r \geq B_\varepsilon$ and $\vartheta \in \mathbb{S}^{2d-1}$. Then,*

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\sigma(\lambda)} = \frac{\pi}{(2\pi)^{d+1}d} \int_{\mathbb{S}^{2d-1}} \frac{d\vartheta}{(\Phi(\vartheta))^{2d}}.$$

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Concerning the eigenvalues:

$$\lambda_j = f\left(\gamma j^{\frac{1}{2d}}(1+o(1))\right), \quad j \to \infty,$$

with

$$\gamma = \sqrt{2\pi}\left(\frac{2d}{\int_{\mathbb{S}^{2d-1}} \frac{d\vartheta}{(\Phi(\vartheta))^{2d}}}\right)^{\frac{1}{2d}},$$

and, for each $h' < \gamma < h$,

$$\lim_{j\to\infty} \frac{\lambda_j}{f(h' j^{\frac{1}{2d}})} = \infty \quad \text{and} \quad \lim_{j\to\infty} \frac{\lambda_j}{f(h j^{\frac{1}{2d}})} = 0.$$

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## Theorem

*Let* $a \in \Gamma_{\rho}^{\infty}(\mathbb{R}^{2d})$ *be hypoelliptic (in the* $\Gamma_{\rho}^{\infty}$*-sense). Suppose that*

$$\lim_{y\to\infty} \frac{yf'(y)}{f(y)} = \beta \in (0,\infty) \quad \textit{exists.}$$

*If*

$$\lim_{r\to\infty} \frac{a(r\vartheta)}{f(r)} = \Phi(\vartheta) > 0$$

*exists uniformly on* $\vartheta \in \mathbb{S}^{2d-1}$*, then*

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# Weyl formula

**Corollary**

*Let a satisfy the assumptions of any of the previous two theorems. Then, in both cases*

$$N(\lambda) \sim \frac{1}{(2\pi)^d} \int_{a(w)<\lambda} dw, \quad \lambda \to \infty.$$

# The assumption $yf'(y)/f(y) \to \beta \in (0, \infty]$

The condition

$$\lim_{y\to\infty} \frac{yf'(y)}{f(y)} \to \beta \in (0, \infty]$$

is related to the (multiplicative) variation of $f$.

- If $\beta < \infty$, then $f$ is regularly varying (in the sense of Karamata) of index $\beta$, that is,

$$\lim_{y\to\infty} \frac{f(\lambda y)}{f(y)} = \lambda^{\beta}, \quad \text{for each } \lambda > 0.$$

Examples: $f(y) = y^{\beta}$, $f(y) = y^{\beta}(\ln y)^{\alpha}$, $f(y) = y^{\beta}(\ln y)^{\alpha}(\ln \ln y)^{\gamma}, \dots$.

- If $\beta = \infty$, then $f$ is rapidly varying (in the sense of de Haan),

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Examples: $f(y) = e^{y^s}$, $s > 0$, $f(y) = e^{M(y)}$, with $M$ associated function of a sequence $M_p \to \infty$.

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The condition

$$\lim_{y\to\infty} \frac{yf'(y)}{f(y)} \to \beta \in (0, \infty]$$

is related to the (multiplicative) variation of $f$.

- If $\beta < \infty$, then $f$ is regularly varying (in the sense of Karamata) of index $\beta$, that is,

$$\lim_{y\to\infty} \frac{f(\lambda y)}{f(y)} = \lambda^{\beta}, \quad \text{for each } \lambda > 0.$$

Examples: $f(y) = y^{\beta}$, $f(y) = y^{\beta}(\ln y)^{\alpha}$, $f(y) = y^{\beta}(\ln y)^{\alpha}(\ln \ln y)^{\gamma}, \dots$.

- If $\beta = \infty$, then $f$ is rapidly varying (in the sense of de Haan),

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# Example

Consider the symbol

$$a(w) = e^{\langle w \rangle^{1/s}} + b(w),$$

where $s > 1$ and $b$ satisfies: $\forall h' > 0$, $\exists C' > 0$ such that

$$|D^\alpha b(w)| \leq C' h'^{|\alpha|} (\alpha!)^\nu e^{\langle w \rangle^{1/s}} \langle w \rangle^{-\rho(|\alpha|+1)}, \ \forall w \in \mathbb{R}^{2d}, \forall \alpha \in \mathbb{N}^{2d},$$

with $\nu < s$ and $s \geq 1/(1-\rho)$.

If we choose $1 < \nu < l < s$ and $\nu/l \leq 1 - 1/s$ and $\nu/l \leq \rho \leq 1 - 1/s$, then one can show that $a \in \Gamma^{(p!^l),\rho}_{p!^\nu}(\mathbb{R}^{2d})$ is hypoelliptic.
Moreover,

$$C_1 e^{|w|^{1/s}} \leq a(w) \leq C_2 e^{|w|^{1/s}}, \ \text{for large } |w|.$$

Hence, our theorem delivers the spectral asymptotics

$$N(\lambda) \sim \frac{(\ln \lambda)^{2ds}}{2^d d!}, \quad \lambda \to \infty,$$

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# Examples: Power series of Shubin polynomials

Let $a(w) = \sum_{|\gamma| \leq m} c_\gamma w^\gamma$ be real-valued elliptic Shubin polynomial of degree $m \geq 2$ such that $a(w) > 0$ for $|w| \gg 1$.

Denote as $a'(w) = \sum_{|\gamma| = m} c_\gamma w^\gamma$ its principal part.

We consider an entire function $P : \mathbb{R} \to \mathbb{R}$

$$P(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{\widehat{M}_n},$$

where $\widehat{M}_n$ is a sequence of positive numbers for which there exists $C_0 \geq 1$ such that

$$C_0^{n-k} \frac{\widehat{M}_n}{(nm)!^s} \geq \frac{\widehat{M}_k}{(km)!^s}, \quad \forall n, k \in \mathbb{N}, \quad \text{with } n \geq k,$$

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# Power series of Shubin elliptic polynomials

Let $s > 1/(1-\rho)$ and $M_p \subset p!^s$ in the $(M_p)$ case and $M_p \prec p!^s$ in the $\{M_p\}$.

**Theorem**

1. *The series* $P(a(w)) = 1 + \sum_{n=1}^{\infty} \frac{(a(w))^n}{\widehat{M}_n}$ *absolutely converges in* $\Gamma^{*,\infty}_{A_p,1}(\mathbb{R}^{2d})$ *and the symbol* $P \circ a$ *is actually hypoelliptic.*
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# Power series of elliptic operators: Spectral asymptotics

Retain the assumptions on $P$, $a$, and $b$. The principal part of $a$ is $a'$.

**Theorem**

*Let $N_1$ and $N_2$ be the spectral counting functions of*

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*Denote as $\{\lambda_j^{(i)}\}_{j\in\mathbb{N}}$ their sequences of eigenvalues, $i = 1, 2$. Then,*

$$N_i(\lambda) \sim c \cdot (P^{-1}(\lambda))^{\frac{2d}{m}} \quad \text{and} \quad \lambda_j^{(i)} = P\left((j/c)^{\frac{m}{2d}}(1 + o(1))\right)$$

*where*

$$c = \frac{\pi}{(2\pi)^{d+1} d} \int_{\mathbb{S}^{2d-1}} \frac{d\vartheta}{(a'(\vartheta))^{\frac{2d}{m}}}.$$

*If in addition $\widehat{M}_n$ is log-convex,*

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# Example

Let the symbols $a, a'$ and the parameters $s, \rho$ be as before. Consider

$$P(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n^{snm}}.$$

Applying the previous theorem, $A_1$ and $A_2$ are $\Gamma^{*,\infty}_{A_p,\rho}$-hypoelliptic pseudo-differential operators. Notice that

$$e^{-sm} \exp\left(\frac{sm\, y^{\frac{1}{sm}}}{e}\right) \leq \exp\left(\widehat{M}(y)\right) \leq e^{sm} \exp\left(\frac{sm\, y^{\frac{1}{sm}}}{e}\right), \quad y \gg 1,$$

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$$\widehat{M}^{-1}(\ln \lambda) \sim \left(\frac{e \ln \lambda}{sm}\right)^{sm}, \quad \lambda \to \infty.$$

Combining these two facts with the spectral asymptotic formulas,

$$N_i(\lambda) \sim \frac{e^{2ds} c}{(sm)^{2ds}} (\ln \lambda)^{2ds} \text{ and } \lambda_j^{(i)} = \exp\left(\frac{sm}{e}\left(\frac{j}{c}\right)^{\frac{1}{2ds}} (1 + o(1))\right),$$

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# Shubin-Sobolev type spaces

As an application of the developed spectral analysis:

- We introduced a new class of infinite order Shubin-Sobolev type spaces.
- This scale of Shubin-Sobolev spaces leads to regularity results for solutions to elliptic infinite order pseudo-differential equations.

For details, see:

S. Pilipović, B. Prangoski, J. Vindas, *Weyl asymptotic formulae and Sobolev spaces for infinite order pseudo-differential operators,* preprint, arXiv:1701.07907.