Pointwise behavior of fractional integrals of modular forms via complex analysis

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Pointwise regularity of fractional integral of modular forms

- Recently (2019), Pastor has found the pointwise Hölder exponent (at every point!) of fractional integrals of modular forms. This covers certain Fourier series

$$g_a(x) = \sum_{n=1}^{\infty} \frac{c_n}{n^a} e^{\frac{2\pi i x}{m}}, \qquad m \in \mathbb{N}.$$

- His arguments are based on approximative functional equations and Tauberian/Abelian theorems for wavelet transforms.

Our goal: To sketch an alternative method for the analysis of irrational points, using basic complex analysis instead of wavelet analysis. (Collaborative work with Frederik Broucke.)

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Riemann's function

According to an account of Weierstrass, Riemann would have suggested

$$R(x) = \sum_{n=1}^{\infty} \frac{e^{in^2\pi x}}{n^2}$$

as an example of a nowhere differentiable function.

- In 1916 Hardy was able to show that R is not differentiable at:

$$\text{irrationals, rationals of the forms } \frac{2r+1}{2s}, \text{ and } \frac{2r}{4s+1}.$$

- Gerver showed in 1970-1971 that R is in turn only differentiable at every rational that is the quotient of two odd integers.
- Smith (1972) and Itatsu (1981) gave simpler treatments of rational points, which (essentially) yielded the pointwise Hölder exponents.
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The Jacobi theta function

Any analysis of Riemann's function passes through the Jacobi theta function:

$$\theta(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}, \qquad \operatorname{Im} z > 0,$$

$$R'(z) = \frac{i\pi}{2}(\theta(z) - 1).$$

θ is modular form of 'weight' $1/2$, satisfies the transformation laws:

$$\theta(z+2) = \theta(z) \quad \text{and} \quad \theta\Big(-\frac{1}{z}\Big) = e^{-\frac{i\pi}{4}}\sqrt{z}\cdot\theta(z).$$

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Modular forms

Let Γ be a subgroup of finite index of $\mathrm{SL}(2,\mathbb{Z})$. A modular form of weight r is a holomorphic function g on the upper-half plane such that

$$g(\gamma z) = \mu_\gamma \cdot (cz+d)^r g(z), \qquad \text{for each } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad |\mu_\gamma| = 1,$$

and such that $g(z) \ll (\operatorname{Im} z)^{-\nu}$ as $\operatorname{Im} z \to 0^+$ for some ν.

Let m be the order of the stabilizer of ∞ in $\mathrm{SL}(2,\mathbb{Z}) \bmod \Gamma$.

- There is $0 \leq \kappa < 1$ such that $g(mz) = \sum_{n=0}^{\infty} c_n e^{2\pi i (n+\kappa) z}$.
- $g(\infty) = \lim_{\operatorname{Im} z \to \infty} g(z)$ and call g cuspidal at ∞ if $g(\infty) = 0$.
- We say that g is cuspidal at $t \in \mathbb{Q}$ if $\dfrac{g(\gamma z)}{(cz+d)^r}$ is cuspidal at ∞, where $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ is such that $\gamma(\infty) = t$.

- Cusp form: if cuspidal at every element of $\mathbb{Q} \cup \{\infty\}$.
- Non-cusp form: otherwise.

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Fractional integrals of modular functions

We assume w.l.o.g. that $m = 1$, so that $g(z) = \sum_{n=0}^{\infty} c_n e^{2\pi i(n+\kappa)z}$.

$$g_a(z) = \frac{1}{(2\pi i)^a} \sum_{n+\kappa>0}^{\infty} \frac{c_n}{(n+\kappa)^a} e^{2\pi i(n+\kappa)z}, \qquad \operatorname{Im} z \geq 0.$$

- Non-cusp forms: uniformely convergent for $a > r$.
- Cusp forms: uniformely convergent for $a > r/2$.

We write $\alpha_a(x)$ for the pointwise Hölder exponent of g_a at x.

Theorem

Let $x \in \mathbb{Q}$.

1. *If g is cuspidal at x, then $\alpha_a(x) = 2a - r$.*
2. *If g is not cuspidal at x, then $\alpha_a(x) = a - r$.*

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Behavior of g_a at irrational numbers

Let ρ be irrational. Let $\tau(\rho) = 2$ if g is a cusp form, otherwise

$$\tau(\rho) = \sup\left\{\tau : \left|\rho - \frac{p}{q}\right| \ll \frac{1}{q^{\tau}} \text{ for infinitely many noncuspidal } \frac{p}{q}\right\}.$$

Theorem

If ρ is irrational, $\boxed{\alpha_a(\rho) = a - r\left(1 - \frac{1}{\tau(\rho)}\right)}$.

- We sketch a proof in the more difficult non-cusp form case.
- For simplicity, we impose some restrictions in the parameters.

Main tool: boundary behavior of g at ρ

1. $g(\rho + iy) \gg y^{-r + \frac{r}{\tau(\rho)} + \varepsilon}$, infinitely often as $y \to 0^+$.
2. $g(\rho + z) \ll y^{\frac{r}{\tau(\rho)} - \varepsilon - r} + y^{-r}|z|^{\frac{r}{\tau(\rho)} - \varepsilon}$, for $0 < y < 1/2$.

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$$\tau(\rho) = \sup\left\{\tau : \left|\rho - \frac{p}{q}\right| \ll \frac{1}{q^\tau} \text{ for infinitely many noncuspidal } \frac{p}{q}\right\}.$$

Theorem

If ρ is irrational, $\boxed{\alpha_a(\rho) = a - r\left(1 - \dfrac{1}{\tau(\rho)}\right)}$.

- We sketch a proof in the more difficult non-cusp form case.
- For simplicity, we impose some restrictions in the parameters.

Main tool: boundary behavior of g at ρ

1. $g(\rho + \mathrm{i}y) \gg y^{-r+\frac{r}{\tau(\rho)}+\varepsilon}$, infinitely often as $y \to 0^+$.
2. $g(\rho + z) \ll y^{\frac{r}{\tau(\rho)}-\varepsilon-r} + y^{-r}|z|^{\frac{r}{\tau(\rho)}-\varepsilon}$, for $0 < y < 1/2$.

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$$g_1(\rho + h) - g_1(\rho) = -h \cdot g(\infty) + \lim_{y \to 0^+} \int_{iy}^{h+iy} g(\rho + z)\, dz.$$

By Cauchy's theorem, the limit of this integral equals

$$\int_0^{i|h|} g(\rho + z)\, dz + \int_{i|h|}^{h+i|h|} g(\rho + z)\, dz - \int_h^{h+i|h|} g(\rho + z)\, dz =: I_1 + I_2 + I_3.$$

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- Thus, $g(z) \ll 1 + |z - \rho|^{\beta - 1}$. Comparing with (1), $\beta \leq 1 - r + \frac{r}{\tau(\rho)}$

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Multifractal spectrum

Using the knowledge of exact pointwise Hölder exponent and a variant of the Jarnik's theorem, Pastor proved:

Theorem

Let $d_a(\alpha)$ be the Hausdorff dimension of $\{x : \alpha_a(x) = \alpha\}$. Then,

1. *If g is a cusp form,*

$$d_a(\alpha) = \begin{cases} 1 & \text{if } \alpha = a - r/2 \\ 0 & \text{if } \alpha = 2a - r \\ -\infty & \text{otherwise.} \end{cases}$$

2. *If g is not a cusp form,*

$$d_a(\alpha) = \begin{cases} 2\left(1 + \frac{\alpha - a}{r}\right) & \text{if } a - r \leq \alpha \leq a - r/2 \\ 0 & \text{if } \alpha = 2a - r \\ -\infty & \text{otherwise.} \end{cases}$$

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Reference

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- F. Broucke, J. Vindas, *The pointwise behavior of Riemann's function,* J. Fractal Geom. 10 (2023)

Some other references:

- F. Chamizo, *Automorphic forms and differentiability properties,* Trans. Amer. Math. Soc. 356 (2004), 1909–1935,
- F. Chamizo, I. Petrykiewicz, S. Ruiz-Cabello, *The Hölder exponent of some Fourier series,* J. Fourier Anal. Appl. 23 (2017), 758–777.
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