The pointwise behavior of Riemann's function

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History of Riemann's function

According to an account of Weierstrass, Riemann would have suggested

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 \pi x)}{n^2} \tag{1}$$

as an example of a nowhere differentiable function.

- Weierstrass could not show that claim, but gave his own example

$$W(x) = \sum_{n=1}^{\infty} a^n \sin(b^n \pi x), \quad 0 < a < 1. \tag{2}$$

- In 1916 Hardy completed the analysis of (2).
- In the same paper, Hardy was able to show that (1) is not differentiable at the following points:
  - irrationals;
  - rationals of the forms $\frac{2r+1}{2s}$ and rationals $\frac{2r}{4s+1}$.

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More history

- Hardy's results seemed to confirm the non-differentiability belief.
- It was then a surprise when Gerver showed in 1970-1971 that Riemann's function is in turn differentiable at every rational that is the quotient of two odd integers, and that it is not differentiable elsewhere.
- Gerver proof is elementary, but long and difficult to grasp.
- Smith (1972) and Itatsu (1981) gave independently simpler treatments of all rational points.
- They both use the Poisson summation formula, i.e.,

$$\sum_{n\in\mathbb{Z}} g(n) = \sum_{n\in\mathbb{Z}} \widehat{g}(n),$$

where we fix the Fourier transform as

$$\widehat{g}(u) = \int_{-\infty}^{\infty} g(x) e(-\mathrm{i}xu) \mathrm{d}\, x \qquad (e(t) = e^{2\pi \mathrm{i} t})$$

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Pointwise Hölder exponent

- Both Smith and Itatsu determined asymptotic estimates describing more detail of the behavior of Riemann's function at rational points.
- Their results (essentially) yield the pointwise Hölder exponent of Riemann's function at rationals.
- This left open the determination of the pointwise Hölder exponents at irrational points.
- Duistermaat (1991) exhibited upper bounds for Hölder exponents at irrationals in terms of diophantine approximation properties of the point.
- Jaffard finally settled the problem in 1996, when he showed that Duistermaat's upper bound was actually the Hölder exponent at every irrational.

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We will sketch a new and simple method to compute the pointwise Hölder exponent of Riemann's function at every point.

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Some words about the idea of our method

- We work with $\phi(z) = \sum_{n=1}^{\infty} \frac{1}{2\pi i n^2} e(n^2 z)$.
- We will compute $\alpha(x) = \sup\{\alpha > 0 \mid \phi(x+h) = \phi(x) + O_x(|h|^{\alpha})\}$.
- Restricting the complex variable z to the upper half-plane, one has

$$\phi'(z) = \frac{1}{2}(\theta(z) - 1), \quad \text{where } \theta(z) = \sum_{n \in \mathbb{Z}} e(n^2 z).$$

- We obtain the basic identity

$$\phi(x+h) - \phi(x) = -\frac{1}{2}h + \frac{1}{2}\lim_{y \to 0^+} \int_{iy}^{h+iy} \theta(x+z)\,dz. \qquad (3)$$

- x rational: we apply the Poisson summation formula to $\theta(x+z)$.
- x irrational: bounds for $\theta(x+z)$ follow from those at rationals.
- The final key step is to use use Cauchy theorem to transform (3):

$$\phi(x+h) - \phi(x) + \frac{1}{2}h = -\frac{1}{2}\int_{\Gamma} \theta(x+z)\,dz.$$

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Number theoretic preliminaries: Gauss sums

Quadratic Gauss sums and generalized quadratic Gauss sums

Let q, p, m be integers with $(p,q) = 1$. These sums are defined as

$$S(q,p) = \sum_{j=1}^{q} e\left(\frac{pj^2}{q}\right) \quad \text{and} \quad S(q,p,m) = \sum_{j=1}^{q} e\left(\frac{pj^2 + mj}{q}\right).$$

The quadratic Gauss sums were evaluated by Gauss himself:

$$S(q,p) = \begin{cases} \varepsilon_q \left(\frac{p}{q}\right)\sqrt{q} & \text{if } q \text{ is odd,} \\ 0 & \text{if } q \equiv 2 \mod 4, \\ (1+\mathrm{i})\overline{\varepsilon_p}\left(\frac{q}{p}\right)\sqrt{q} & \text{if } q \equiv 0 \mod 4. \end{cases}$$

$\left(\frac{p}{q}\right)$ is the Jacobi symbol and (n odd) $\varepsilon_n = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 4, \\ \mathrm{i} & \text{if } n \equiv 3 \bmod 4. \end{cases}$

Elementary manipulations lead to the bounds:

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Let q, p, m be integers with $(p, q) = 1$. These sums are defined as

$$S(q,p)=\sum_{j=1}^{q} e\left(\frac{pj^2}{q}\right) \quad \text{and} \quad S(q,p,m)=\sum_{j=1}^{q} e\left(\frac{pj^2+mj}{q}\right).$$

The quadratic Gauss sums were evaluated by Gauss himself:

$$S(q,p)=\begin{cases}\varepsilon_q\left(\frac{p}{q}\right)\sqrt{q} & \text{if } q \text{ is odd},\\ 0 & \text{if } q\equiv 2 \mod 4,\\ (1+\mathrm{i})\overline{\varepsilon_p}\left(\frac{q}{p}\right)\sqrt{q} & \text{if } q\equiv 0 \mod 4.\end{cases}$$

$\left(\frac{p}{q}\right)$ is the Jacobi symbol and (n odd) $\varepsilon_n=\begin{cases}1 & \text{if } n\equiv 1 \bmod 4,\\ \mathrm{i} & \text{if } n\equiv 3 \bmod 4.\end{cases}$

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Lemma

Suppose $(p,q)=1$ and $y=\operatorname{Im} z>0$. Then

$$\theta\left(\frac{p}{q}+z\right)=\frac{e^{\pi i/4}}{q\sqrt{2}}z^{-1/2}\left(S(q,p)+2\sum_{m=1}^{\infty}S(q,p,m)\exp\left(-\frac{i\pi m^2}{2q^2z}\right)\right).$$

Proof. Note that $e(pn^2/q)$ is q-periodic in n, writing $n=j+mq$,

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The behavior at the rationals: behavior of θ

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The behavior at the rationals: behavior of ϕ

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$$\phi_{q,p}(z)=\sum_{m=1}^{\infty}\frac{S(q,p,m)}{2\pi i m^2}e(m^2 z)\ll\sqrt{q}$$

For $y>0$,

$$\phi\left(\frac{p}{q}+h+iy\right)=\phi\left(\frac{p}{q}+iy\right)-\frac{1}{2}h+\frac{1}{2}\int_{iy}^{h+iy}\theta\left(\frac{p}{q}+\zeta\right)d\zeta,$$ and very last term is

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The behavior at the rationals of Re ϕ

TABLE 1. Behavior of $\mathrm{Re}\big(\phi(p/q+h)-\phi(p/q)\big)$

q mod 4	p mod 4	$h<0$	$h>0$
1	any	$-\left(\frac{p}{q}\right)\frac{1}{2\sqrt{q}}\sqrt{\lvert h\rvert}+O_q(\lvert h\rvert)$	$\left(\frac{p}{q}\right)\frac{1}{2\sqrt{q}}\sqrt{h}+O_q(h)$
3	any	$-\left(\frac{p}{q}\right)\frac{1}{2\sqrt{q}}\sqrt{\lvert h\rvert}+O_q(\lvert h\rvert)$	$-\left(\frac{p}{q}\right)\frac{1}{2\sqrt{q}}\sqrt{h}+O_q(h)$
2	any	$-\frac{1}{2}h+O\big(q^{3/2}\lvert h\rvert^{3/2}\big)$	$-\frac{1}{2}h+O\big(q^{3/2}h^{3/2}\big)$
0	1	$-\left(\frac{q}{p}\right)\frac{1}{\sqrt{q}}\sqrt{\lvert h\rvert}+O_q(\lvert h\rvert)$	$-\frac{1}{2}h+O\big(q^{3/2}h^{3/2}\big)$
0	3	$-\frac{1}{2}h+O\big(q^{3/2}\lvert h\rvert^{3/2}\big)$	$\left(\frac{q}{p}\right)\frac{1}{\sqrt{q}}\sqrt{h}+O_q(h)$

Corollary

If p and q are both odd, then $f(x)=\sum_{n=1}^{\infty}n^{-2}\sin(\pi n^2 x)$ is differentiable at $x=p/q$; otherwise the Hölder exponent of f at r equals $1/2$.

The behavior at the rational of Re ϕ

TABLE 1. Behavior of $\mathrm{Re}\big(\phi(p/q+h)-\phi(p/q)\big)$

q mod 4	p mod 4	$h<0$	$h>0$
1	any	$-\left(\frac{p}{q}\right)\frac{1}{2\sqrt{q}}\sqrt{\lvert h\rvert}+O_q(\lvert h\rvert)$	$\left(\frac{p}{q}\right)\frac{1}{2\sqrt{q}}\sqrt{h}+O_q(h)$
3	any	$-\left(\frac{p}{q}\right)\frac{1}{2\sqrt{q}}\sqrt{\lvert h\rvert}+O_q(\lvert h\rvert)$	$-\left(\frac{p}{q}\right)\frac{1}{2\sqrt{q}}\sqrt{h}+O_q(h)$
2	any	$-\frac{1}{2}h+O\big(q^{3/2}\lvert h\rvert^{3/2}\big)$	$-\frac{1}{2}h+O\big(q^{3/2}h^{3/2}\big)$
0	1	$-\left(\frac{q}{p}\right)\frac{1}{\sqrt{q}}\sqrt{\lvert h\rvert}+O_q(\lvert h\rvert)$	$-\frac{1}{2}h+O\big(q^{3/2}h^{3/2}\big)$
0	3	$-\frac{1}{2}h+O\big(q^{3/2}\lvert h\rvert^{3/2}\big)$	$\left(\frac{q}{p}\right)\frac{1}{\sqrt{q}}\sqrt{h}+O_q(h)$

Corollary

If p and q are both odd, then $f(x)=\sum_{n=1}^{\infty}n^{-2}\sin(\pi n^2 x)$ is differentiable at $x=p/q$; otherwise the Hölder exponent of f at r equals $1/2$.

The behavior at irrational ρ: preparations

- Hardy result gives the upper bound $\alpha(x) \le 3/4$.
- Duistermaat improved upon this result refining the use of rational approximations.
- Irrational has continued fraction $\rho = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots}}$
- Its nth convergent is $r_n = \frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$
- We define τ_n via

$$|\rho - r_n| = \left(\frac{1}{q_n}\right)^{\tau_n}.$$

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$|\rho - r_n| = \left(\frac{1}{q_n}\right)^{\tau_n}$ and $\tau(\rho) := \limsup_{k\to\infty} \tau_{n_k}$

Exploiting the uniform bounds

$$\phi(p/q + h) = \phi(p/q) + C^{-}_{p/q}|h|_{-}^{1/2} + C^{+}_{p/q}|h|_{+}^{1/2} - h/2 + O(q^{3/2}|h|^{3/2})$$

Duistermaat found that given $\varepsilon > 0$, there is a sequence (h_j) such that

$$|\phi(\rho + h_j) - \phi(\rho)| \geq c_{\varepsilon,\rho}\left|h_j\right|^{\frac{1}{2}+\frac{1}{2(\tau(\rho)-\varepsilon)}}, \quad h_j \to 0.$$

This implies

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$$\alpha(\rho) \leq 1/2 + 1/(2\tau(\rho)).$$

The same holds for the Hölder exponent of $\operatorname{Re}\phi$ and $\operatorname{Im}\phi$.

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The behavior at the irrationals: Jaffard's theorem

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Our proof uses the following bound on the θ function.

Proposition

Suppose $z = x + iy$ with $y > 0$. For each $\varepsilon > 0$,

$$\theta(\rho + z) \ll |z|^{\frac{1}{2\tau(\rho)} - \varepsilon - \frac{1}{2}} + y^{-1/2}|z|^{\frac{1}{2\tau(\rho)} - \varepsilon} \qquad (|z| \ll 1) \qquad (4)$$

The bound (4) is due to Jaffard, we gave a much simpler proof based on

$$\theta(p/q + \zeta) \ll q|\zeta|^{-1/2}|S(q,p)| + \sqrt{q}|\zeta|^{1/2}/(\operatorname{Im}\zeta)^{1/2},$$

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Our proof of the lower bound: $\alpha(\rho) \geq \frac{1}{2} + \frac{1}{2\tau(\rho)}$

We use the bound

$$\phi(\rho + h) - \phi(\rho) = -\frac{1}{2}h + \frac{1}{2} \lim_{y \to 0^+} \int_{\mathrm{i}y}^{h+\mathrm{i}y} \theta(\rho + z)\,\mathrm{d}z.$$

By Cauchy's theorem, the limit of this integral equals

$$\int_0^{\mathrm{i}|h|} \theta(\rho+z)\,\mathrm{d}z + \int_{\mathrm{i}|h|}^{h+\mathrm{i}|h|} \theta(\rho+z)\,\mathrm{d}z - \int_h^{h+\mathrm{i}|h|} \theta(\rho+z)\,\mathrm{d}z =: I_1 + I_2 + I_3.$$

Using the bounds $\theta(\rho + z) \ll |z|^{\frac{1}{2\tau(\rho)} - \varepsilon - \frac{1}{2}} + y^{-1/2}|z|^{\frac{1}{2\tau(\rho)} - \varepsilon}$, we get

$$I_1 \ll \int_0^{|h|} y^{-\frac{1}{2} + \frac{1}{2\tau(\rho)} - \varepsilon}\,\mathrm{d}y \ll |h|^{\frac{1}{2} + \frac{1}{2\tau(\rho)} - \varepsilon},$$

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Reference

The talk is based on the following collaborative preprint with Frederik Broucke:

F. Broucke, J. Vindas, *The pointwise behavior of Riemann's function*, arXiv:2109.08499