

On the density hypothesis for L-functions associated with holomorphic cusp forms.

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□ The Riemann hypothesis: The distribution of prime numbers is intimately connected with the properties of the Riemann zeta function, defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \operatorname{Re} s > 1,$$

and extended by analytic continuation as a meromorphic function with a single simple pole at $s=1$, so that

$$\zeta(s) - \frac{1}{s-1} \text{ is entire.}$$

The Riemann hypothesis claims that the only zeros of ζ are located at:

① $s = -2, -4, -6, \dots$ (trivial zeros)

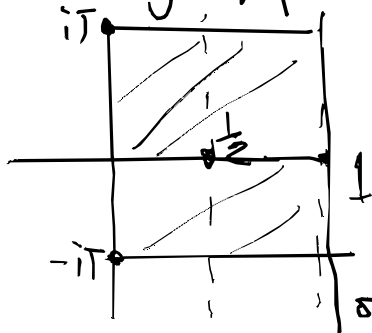
② $\operatorname{Re} s = \frac{1}{2}$.

The claim ① follows from the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s),$$

while $\{s \in \mathbb{C} : \operatorname{Re} s = \frac{1}{2}\}$ is the so-called critical line. It was shown by Hardy that ζ has infinitely many zeros on the critical line. ①

many zeros on this critical line. In fact,
it is even known that the number of zeros $N(T)$ of ζ
lying on the critical strip $0 < \text{Re } s < 1$ and having
Imaginary part $|\text{Im } s| \leq T$.



$$\text{is } N(T) = \frac{T}{\pi} \log \frac{T}{2\pi e} + O(\log T),$$

$T \rightarrow \infty$,

a formula stated by Riemann (1859) and
shown by von Mangoldt (1905).

The RH is one of the greatest unsolved* problems in
mathematics. It is equivalent to the following quantified
form of the PNT. Let $\pi(x)$ be the number of
prime numbers $\leq x$. Then RH holds $\iff$

for each $\varepsilon > 0$,

$$\pi(x) = \int_2^x \frac{du}{\log u} + O_\varepsilon(x^{\frac{1}{2}-\varepsilon}).$$

Here are some of the records for zero free
regions of ζ

* David Hilbert once said: If I would wake up
after sleeping 1000 years, my first question would
be: has RH already been proved? (2)

zero free region $s = \sigma + it$.	Record
$\sigma \geq 1 - \frac{c}{\log(1+ t)}$	de la Vallée-Poussin 1897
$\sigma \geq 1 - \frac{\log \log(e+ t)}{\log(1+ t)}$	Littlewood 1922
$\sigma \geq 1 - \frac{c}{(\log(1+ t))^{2/3} (\log \log(e+ t))^{1/3}}$	Vinogradov-Korobov 1958 (best known)

[2] Zero-density estimates: There is also a great interest in zero-density estimates. Denote as $N(\sigma, T)$ the number of zeros of ζ on the rectangle $|t| \leq T$ and $\sigma \leq \Re s \leq 1$.



In 1937, Ingham connected estimates of the form

$$(1) \quad N(\sigma, T) \ll_{\varepsilon} T^{c(1-\sigma)+\varepsilon}, \quad \frac{1}{2} \leq \sigma \leq 1,$$

(3)

with the behavior of primes in short intervals. Using
(1) and the so-called exact von Mangoldt formula**

$$\sum_{x \leq p \leq x+h} \log p \sim h, \quad x \rightarrow \infty, \quad h \gg x^{1+\varepsilon - \frac{1}{C}}$$

This estimate when $C=2$ is of the same quality as if one would assume the RH.

Since this is the case for several other arithmetic results, the (conjectural) inequality

$$(DH) \quad N(\sigma, T) \ll_{\varepsilon} T^{2(1-\sigma)+\varepsilon} \quad \frac{1}{2} \leq \sigma \leq 1$$

is known as the density hypothesis.

[3] The density hypothesis for the Riemann zeta function.

The (DH) is out of reach of current methods. Nevertheless there has been a lot of progress maximizing its range of validity. Let $\sigma_0 \geq \frac{1}{2}$ be such that

$$N(\sigma, T) \ll_{\varepsilon} T^{2(1-\sigma)+\varepsilon} \quad \sigma_0 \leq \sigma \leq 1.$$

We have the following results:

** If ψ is the Chebyshev function,

$$\psi(x) = x - \sum_{\substack{\chi(p) \neq 0 \\ \text{Im } p > 0}} \frac{\chi(p)}{p} - \log(2\pi) - \frac{1}{2} \log\left(1 - \frac{1}{x^2}\right) \quad (4)$$

τ_0	Record
0.9	Montgomery, 1969
0.8333...	Huxley, 1972
0.8076...	Romachondia, 1975
0.8	Huxley, 1975
$11/14 = 0.7857...$	Jutila, 1977
$25/32 = 0.78125$	Bourgain, 2000

4] Dirichlet L-functions. Let χ be a Dirichlet character mod q , that is,

$\chi: \mathbb{N} \rightarrow \mathbb{C}$ such that

1. χ is completely multiplicative
2. χ is q -periodic
3. $\chi(n) = 0 \Leftrightarrow (n, q) \neq 1$.

It is associated L-function is

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Let $N_{\chi}(\sigma, T)$ be the number of zeros in $[\sigma, \sigma+1] + i[-T, T]$.

The density hypothesis estimate

$$(2) \quad \sum_{\chi \bmod q} N_{\chi}(\sigma, T) \ll (qT)^{2(1-\sigma)+\varepsilon}, \quad \sigma_0 \leq \sigma \leq 1$$

has also been very much studied for L-functions

The best record is due to Heath-Brown (1979)

Theorem (2) holds with $\sigma_0 = 15/19 = 0.7894...$

We note that Bourgain (2002) applied his so-called "dichotomy method (developed to get his current best record for the Riemann zeta function) to Dirichlet L-functions, but this time he only could recover $\sigma_0 = 15/19$.

[5] L-functions associated with holomorphic cusp forms

(6)

We have recently applied the dichotomy approach to obtain the current record for the range of validity for the density hypothesis estimate for L -functions associated to certain holomorphic cusp forms. We introduce some notations to state our result.

Let $\mathbb{H}$ be the upper half-plane. A holomorphic function $f: \mathbb{H} \rightarrow \mathbb{C}$ is called a holomorphic form of weight $k \in \mathbb{N}$ (w.r.t. the full $SL(2, \mathbb{Z})$) if

① For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$

$$f(\gamma(z)) = (cz + d)^k f(z) ; \gamma(z) = \frac{az + b}{cz + d}.$$

② $f(z) = O(1)$ as $\text{Im } z \rightarrow \infty$.

Taking $\gamma(z) = z + 1$, f is 1-periodic and therefore always has Fourier series expansion

$$f(z) = \sum_{n=0}^{\infty} a_f(n) e^{2\pi i n z}.$$

We denote M_k the space of modular forms of

weight k . The form is called a
 (holomorphic) cusp form if $a_f(0) = 0$,
 we write $f \in M_{k,0}$. We are interested in those
 f for which their coefficients are multiplicative
 functions, i.e.,

$$(3) \quad a_f(n \cdot m) = a_f(m) \cdot a_f(n), \quad (n, m) = 1.$$

In order to characterize this property Hecke
 introduce a family of operators $(n=1, 2, \dots)$

$$T_n: M_k \rightarrow M_k \quad \text{and} \quad T_n: M_{k,0} \rightarrow M_{k,0}$$

such that every simultaneous eigenvector of
 $T_n, n=1, 2, \dots$ satisfies (3). We called
 such forms (Hecke) eigenforms.

Here

$$(T_n f)\left(\frac{z}{d}\right) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2}\right).$$

From now on we fix a holomorphic Hecke
 eigenform of weight $k \in \mathbb{N}$, and we normalize

$$a_f(1) = 1.$$

Lemma: (Deligne^{***}, 1974) $|a_f(n)| \leq n^{\frac{k-1}{2}} d(n)$,
 with $d(n) = \sum_{d|n} 1$. //

We define $\lambda_f(n) = \frac{a_n}{n^{\frac{k-1}{2}}}$ the (normalised)

L-Function associated with f is

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} \left(= \prod_p \frac{1}{\left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right)} \right)$$

Hecke showed that $L(s, f)$ extends as entire function to $\mathbb{C}$, while Good (1982) showed the second moment estimate

$$\int_0^T |L(\tfrac{1}{2} + it, f)|^2 dt \ll_{\varepsilon} T^{1+\varepsilon}.$$

If $N_f(\sigma, T)$ is the number of zeros of f on $[\sigma, 1] + i[-T, T]$, the following summarises what is known for the range of validity for

$$N_f(\sigma, T) \ll_{\varepsilon} T^{2(1-\sigma)+\varepsilon}, \quad \sigma_0 \leq \sigma \leq 1$$

"This was shown in the context of Deligne's proof of Weil conjecture: the Riemann hypothesis for the zeta function of a nonsingular projective algebraic variety over $\mathbb{F}_q$ " (9)

T_0	Record
$53/60 = 0.8833...$	Ivić, 1989
$1407/1601 = 0.8788...$	Chen, Debruyne, V. 2024

Our result [1] actually applies to L-functions

$$L(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s} :$$

- satisfying a Ramanujan conjecture
for its coefficients (i.e., multiplicity,
Euler type identity, no good bounds on coefficients)
- satisfying polynomial bounds on suitable
half-planes.
- satisfying a second moment type estimate $\ll T^{1/2}$.

We have also used the same method to give other
density estimates for the Riemann zeta function,
which according to [2,3] are the best available
ones *** in certain regions.

6 References

- [1] B. Chen, G. Debruyne, J. Vindas, On the density hypothesis for L -functions associated with holomorphic cusp forms, *Rev. Mat. Iberoam.* 40 (2024), 2179-2202.
- [2] T. Trudgian, A. Yang, Toward optimal exponent pairs, *Math. Comp.* 94 (2025), 1467-1502.
- [3] T. Tao, T. Trudgian, A. Yang, New exponent pairs, zero density estimates, and zero additive energy estimates: a systematic approach, preprint, 2025, arXiv: 2501.16779.

*** Our bound is $N(\sigma, T) \ll T^{\frac{24(1-\sigma)+\varepsilon}{80\sigma-11}}$ in the range

$$\frac{279}{314} \approx 0.8885 \leq \sigma < \frac{17}{18} \approx 0.944\dots,$$

which accordy to [2,3] is sharpest available on this range.