Some developments on the Wiener-Ikehara theorem

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8th Workshop on Graph Theory and Applications
Ohrid, August 9, 2024
Linnaeus University
Växjö, August 28, 2024
International Conference on Generalized Functions
Turin, September 17, 2024
Arpilysm 1: Applications of Mathematics to Mathematics
Arpino, November 11, 2024

In this talk we will discuss some developments on the Wiener-Ikehara theorem.

Complex Tauberian theorems have been strikingly useful tools in diverse areas such as:

- Analytic number theory.
- Spectral theory.
- Last four decades: operator theory and semigroups.

Main questions we will address:

1. Relax boundary requirements to a minimum.
2. Exact form (if and only if form).
3. Absence of remainders.
4. Some remainder terms and optimality.

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The classical Wiener-Ikehara theorem

$\mathcal{L}\{S;s\} = \int_0^\infty S(x)e^{-sx}\mathrm{d}x$ and $\mathcal{L}\{\mathrm{d}S;s\} = \int_{0^-}^\infty e^{-sx}\mathrm{d}S(x)$; $s = \sigma + it$.

Theorem (Wiener-Ikehara, Laplace transforms)

Let S be a non-decreasing function (Tauberian hypothesis) such that $\mathcal{L}\{S;s\}$ converges for $\Re e\, s > 1$. If

$$\mathcal{L}\{S;s\} - \frac{A}{s-1} \qquad \left(\text{or equivalently } \mathcal{L}\{\mathrm{d}S;s\} - \frac{A}{s-1}\right)$$

has analytic continuation through $\Re e\, s = 1$, then $S(x) \sim Ae^x$.

Theorem (Wiener-Ikehara, version for Dirichlet series)

Let $a_n \geq 0$. Suppose $\sum_{n=1}^\infty a_n n^{-s}$ converges for $\Re e\, s > 1$. If

$$\sum_{n=1}^\infty \frac{a_n}{n^s} - \frac{A}{s-1}$$

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From the Wiener-Ikehara theorem to the PNT:

The Prime Number Theorem (PNT) asserts that

$$\pi(x) = \sum_{p \leq x} 1 \sim \frac{x}{\log x}$$

- PNT is equivalent to $\psi(x) = \sum_{p^k \leq x} \log p = \sum_{n \leq x} \Lambda(n) \sim x$.
- $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ has analytic continuation to $\mathbb{C}$ except for a simple pole with residue 1 at $s = 1$.
- Logarithmic differentiation of $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ leads to

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}, \quad \Re e\, s > 1.$$

- $(s-1)\zeta(s)$ has no zeros on $\Re e\, s = 1$, so

$$-\frac{d}{ds}(\log((s-1)\zeta(s))) = -\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$$

is analytic in a region containing $\Re e\, s \geq 1$. The rest follows from the Wiener-Ikehara theorem.

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Remarks on the Wiener-Ikehara theorem

- Historically, the Wiener-Ikehara theorem improved upon a Tauberian theorem of Landau (1908) by eliminating the unnecessary hypothesis $G(s) \ll |s|^N$ on

$$G(s) = \mathcal{L}\{S; s\} - \frac{A}{s-1}$$

- The hypothesis $G(s)$ has analytic continuation to $\Re e\, s = 1$ can be significantly relaxed to "good boundary behavior":
  1. G(s) has continuous extension to $\Re e\, s = 1$.
  2. L^1_{loc}-boundary behavior: $\lim_{\sigma \to 1^+} G(\sigma + it) \in L^1(I)$ for every finite interval I.
  3. Local pseudofunction boundary behavior (Korevaar, 2005).
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Pseudofunctions

The concept of pseudofunctions naturally arises in harmonic analysis.

- $C_0(\mathbb{R})$: the space of continuous functions vanishing at $\pm\infty$.
- Pseudofunctions: $PF(\mathbb{R}) = \{g \in \mathcal{S}'(\mathbb{R}) : \widehat{g} \in C_0(\mathbb{R})\}$.

Given an open set $I \subseteq \mathbb{R}$, we define the local space:

- $PF_{loc}(I)$: g such that for all bounded open subinterval $I' \subset I$ there is $f \in PF(\mathbb{R})$ such that $g = f$ on I'.
- $L^1_{loc}(I) \subset PF_{loc}(I)$.

Let G be analytic on $\Re e\, s > 1$ and $I \subset \mathbb{R}$ be open.

We say that G has local pseudofunction boundary behavior on $1 + iI$ if it has distributional boundary values there, i.e.

$$\lim_{\sigma \to 1^+} G(\sigma + it) = g(t) \text{ in } \mathcal{D}'(I)$$

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A Tauberian condition: log-linear slow decrease

Classically, a function is called slowly decreasing in the sense of Schmidt (i.e. in multiplicative form) if

$$\liminf_{\lambda\to 1^+}\liminf_{x\to\infty}\inf_{1\le a\le\lambda}(f(ax)-f(x))\ge 0$$

It is called linearly slowly decreasing if

$$\liminf_{\lambda\to 1^+}\liminf_{x\to\infty}\inf_{1\le a\le\lambda}\frac{f(ax)-f(x)}{x}\ge 0$$

Definition

We call a function S log-linearly slowly decreasing if $S(\log x)$ is linearly slowly decreasing, that is, for each $\varepsilon>0$ there are $\delta, x_0>0$ such that

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holds for all $x\ge x_0$ and $0<h<\delta$.

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Extension of the Korevaar-Wiener-Ikehara theorem

Theorem (Debruyne and V., 2016)

Let $S \in L^1_{loc}[0, \infty)$. Then,

$$S(x) \sim Ae^x$$

if and only if

1. $\mathcal{L}\{S; s\}$ *converges for* $\Re e\, s > 1$,
2. $\mathcal{L}\{S; s\} - \dfrac{A}{s-1}$ *has local pseudofunction boundary behavior on* $\Re e\, s = 1$, *and*
3. *S is log-linearly slowly decreasing.*

- This and related Tauberians have found many recent applications in the theory of Beurling generalized primes.
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Let S be non-decreasing.

If one wishes to attain a stronger relative remainder than $o(1)$, i.e.,

$$\frac{S(x)}{e^x} = A + O(\rho(x)) \qquad \text{with} \qquad \rho(x) = o(1),$$

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It is folklore that remainders can be obtained from:

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In this part of the talk we explore:

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A conjecture under merely analytic continuation

M. Müger raised the question of whether it is still possible to obtain error terms without assuming bounds on the analytic continuation of $\mathcal{L}\{\mathrm{d}S; s\} - A/(s-1)$ to a half-plane. He actually conjectured one could get the following error term:

Conjecture (Müger, 2018)

Let $0 < \alpha < 1$ and $A > 0$. If $\mathcal{L}\{\mathrm{d}S; s\} - A/(s-1)$ has analytic continuation to $\Re e\, s > \alpha$, then

$$S(x) = Ae^x + O_\varepsilon(e^{x(\frac{\alpha+2}{3}+\varepsilon)}), \quad \forall \varepsilon > 0.$$

We refuted this conjecture; in fact:

Negative general answer

No remainder can be expected in the Wiener-Ikehara theorem, even from entire continuation.

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Theorem (Debruyne and V., 2018; Broucke, Debruyne, and V., 2021; Callewaert, Neyt, and V., 2024)

Let ρ be an arbitrary positive function tending to 0. There is a non-decreasing function S on $[0, \infty)$ such that

$$\mathcal{L}\{\mathrm{d}S; s\} = \int_{0^-}^{\infty} e^{-sx} dS(x) \qquad \text{converges for } \Re e\, s > 1$$

and

$$\mathcal{L}\{dS; s\} - \frac{A}{s-1}$$

admits extension to $\mathbb{C}$ as an entire function for some $A > 0$, but such that

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We consider the following situation, with S, M, and K positive non-decreasing

- $\mathcal{L}\{\mathrm{d}S; s\} - \dfrac{A}{s-1}$ has analytic continuation to

$$\Omega_M = \left\{ s = \sigma + it \in \mathbb{C} : |1 - \sigma| \leq \frac{1}{M(|t|)} \right\}.$$

- Bound $\mathcal{L}\{\mathrm{d}S; s\} - \dfrac{A}{s-1} = O(K(|t|))$ for $s = \sigma + it \in \Omega_M$.
- $M_{K,log}(t) = M(t)[\log(t \cdot K(t) \cdot \log t)]$.

Theorem (Debruyne, 2024; improving upon Stahn, 2018)

For any $0 < c < 1$, *we always have*

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With the previous notation:

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Assume that, for $C > 0$,

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for all non-decreasing S such that $\mathcal{L}\{\mathrm{d}S; s\} - \dfrac{A}{s-1}$ analytically extends to Ω_M with bound $O(K(|\Im m\, s|))$ there. Then,

$$\frac{1}{M_K^{-1}(x)} \ll \rho(x)$$

with as before M_K^{-1} the inverse function of $M_K(t) = M(t)[\log(t \cdot K(t))]$.

Optimality of the remainders

With the previous notation:

Theorem (Debruyne, 2024; improving upon Debruyne-Seifert 2019)

Assume that, for $C > 0$,

$$M(x) = O\left(\exp\left(\exp\left(CxK(x)\right)\right)\right).$$

Suppose that ρ is a non-increasing function such that

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Further developments

- (Debruyne, 2024): More general remainder theory; including non-analytic extension hypotheses.
- (Koga 2021; Chen & V., 2024): Conditions merely on real part of Laplace transform with applications in renewal theory.
- Finite form (Graham & Vaaler, 1981): For non-decreasing S, if

$$G(s) = \mathcal{L}\{S; s\} - \frac{A}{s-1}$$

has pseudofunction boundary behavior on $1 + i(-\lambda, \lambda)$, then

$$A\frac{2\pi/\lambda}{e^{2\pi/\lambda}-1} \leq \liminf_{x\to\infty} \frac{S(x)}{e^x} \leq \limsup_{x\to\infty} \frac{S(x)}{e^x} \leq A\frac{2\pi/\lambda}{1-e^{-2\pi/\lambda}}$$

- (Tranoy & V.): Finite forms with milder boundary assumptions.
- (Tenenbaum): Effective Wiener-Ikehara theorem in terms of

$$\eta(\sigma, \lambda) = \int_{-\lambda}^{\lambda} |G(2\sigma + it) - G(\sigma + it)| \mathrm{d}t$$

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- G. Debruyne, A general quantified Ingham-Karamata Tauberian theorem, preprint, 2024.
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Book references on complex Tauberians

- W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander, Vector-valued Laplace transforms and Cauchy problems, 2011 (2nd edition).
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