The absence of remainders in the Wiener-Ikehara theorem

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on the occasion of his 70th Birthday

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The Wiener-Ikehara theorem is a landmark in 20th century analysis. It states,

**Theorem (Wiener-Ikehara)**

Let $S$ be a non-decreasing function and suppose that

$$G(s) := \int_1^{\infty} S(x)x^{-s-1}dx \text{ converges for } \Re e s > 1$$

and that there exists $A$ such that $G(s) - A/(s-1)$ admits a continuous extension to $\Re e s \geq 1$, then

$$S(x) = Ax + o(x).$$  \hspace{1cm} (1)

We discuss here whether it is possible to improve the remainder in (1) under an analytic continuation hypothesis.

We will give a negative answer to a conjecture of M. Müger.

The talk is based on collaborative work with Gregory Debruyne.
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Remainders in the Wiener-Ikehara theorem

$S$ non-decreasing with Mellin transform $G(s)$ on $\Re s > 1$.

If one wishes to attain a stronger remainder than $o(x)$, i.e.,

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it is natural to strengthen the regularity assumptions on

$$G(s) - \frac{A}{s-1}. \quad (2)$$

Our goal: to study the following hypothesis:

(2) has analytic continuation to $\Re s > \alpha$, where $0 < \alpha < 1$.

Well-known: remainders can be obtained if bounds on (2) hold.

Theorem (Simplest example)

If $G(s) - \frac{A}{s-1} \ll (1 + |\Im s|)^{N-1}$ on the strip $\alpha < \Re s < 2$,

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M. Müger raised the question of whether it is still possible to obtain error terms without the bounds on the analytic continuation of $G(s) - a/(s - 1)$. He actually conjectured one could get the error term

**Conjecture (Müger, 2017)**

Let $0 < \alpha < 1$ and $a > 0$. If $G(s) - \frac{a}{s - 1}$ has analytic continuation to $\Re s > \alpha$, then

$$S(x) = ax + O_\varepsilon(x^{\frac{\alpha + 2}{3} + \varepsilon}), \quad \forall \varepsilon > 0.$$ 

We show in this talk that the latter conjecture is false; in fact, we report the following more general result:

**Negative general answer**

No reasonably good remainder can be expected in the Wiener-Ikehara theorem, with solely the analyticity on $\Re s > \alpha$. 

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Wiener-Ikehara theorem
Theorem (Debruyne and V., 2018)

Let $\rho$ be a positive function, $A > 0$, and $0 < \alpha < 1$. Suppose that every non-decreasing function $S$ on $[1, \infty)$, whose Mellin transform $G(s)$ is such that

$$G(s) - \frac{A}{s - 1}$$

admits an analytic extension to $\Re s > \alpha$, satisfies

$$S(x) = Ax + O(x\rho(x)).$$

Then, one must necessarily have

$$\rho(x) = \Omega(1).$$

(the latter means $\rho(x) \nrightarrow 0$.)

The rest of the talk is devoted to outline the proof of this result.
We will use functional analysis and need to make a vector space out of our problem.

- As the "Tauberian theorem hypothesis" holds for some $A > 0$, it holds $\forall A > 0$.

Set $T(x) = S(x) - Ax$.

- The Mellin transform $G_T(s) := \int_1^\infty x^{-1-s} T(x)dx$ has analytic continuation to $\Re s > \alpha$.
- If $T$ is absolutely continuous, $T'(x)$ is bounded from below.
- The asymptotic formula for $S$ becomes $T(x) \ll x\rho(x)$.

We shall use less to show our original result, i.e., it is contained in:

**Theorem**

If $T(x) = O(x\rho(x))$ for any $T$ with $T' \in L^\infty(1, \infty)$ such that $G_T(s)$ has analytic continuation to $\Re s > \alpha$, then

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Wiener-Ikehara theorem
The proof: open mapping theorem argument

If $T(x) = O(x\rho(x))$ for any $T$ with $T' \in L^\infty(1, \infty)$ such that $G_T(s)$ has analytic continuation to $\Re s > \alpha$, then $\rho(x) = \Omega(1)$.

- Let $Y$ be the Fréchet space of Lipschitz continuous functions $T$ on $[1, \infty)$ such that $G_T(s)$ can be analytically continued to $\Re s > \alpha$ and continuously extended to $\Re s \geq \alpha$.
- The natural topology of $Y$ is given by the seminorms
  \[ \|T\|_{Y,n} = \text{ess sup}_{x \geq 1} |T'(x)| + \sup_{\Re s \geq \alpha, |\Im s| \leq n} |G_T(s)|, \quad n = 1, 2, \ldots \]
- The second Fréchet space $Z \subseteq Y$ is defined via the norms
  \[ \|T\|_{Z,n} = \sup_{x \geq 1} \left| \frac{T(x)}{x\rho(x)} \right| + \|T\|_{Y,n}, \quad n = 1, 2, \ldots \]
- The inclusion $Z \to Y$ is continuous and our hypothesis is $Z = Y$.

The open mapping theorem implies there are $N, C > 0$ such that
\[ \sup_{x \geq 1} \left| \frac{T(x)}{x\rho(x)} \right| \leq C \|T\|_{Y,N} \]
If \( T(x) = O(x\rho(x)) \) for any \( T \) with \( T' \in L^\infty(1, \infty) \) such that \( GT(s) \) has analytic continuation to \( \Re s > \alpha \), then \( \rho(x) = \Omega(1) \).

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The proof: using the inequality

The key inequality

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\sup_{x \geq 1} \left| \frac{T(x)}{x \rho(x)} \right| \leq C \left( \text{ess sup}_{x \geq 1} |T'(x)| + \sup_{\Re s \geq \alpha, |\Im m s| \leq N} |G_T(s)| \right)
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extends to the completion of \( Y \) with respect to \( \| \cdot \|_{Y,N} \).

Any \( T \) for which \( T'(x) = o(1), T(1) = 0 \), and whose Mellin transform has analytic continuation in a neighborhood of \( \{ s : \Re s \geq \alpha, |\Im m s| \leq N \} \) is in that completion.

What remains to be done?

- We further proceed by contradiction and assume that \( \rho(x) \to 0 \).
- We construct a \( T \) with these properties such that when inserted in the key inequality contradicts \( \rho(x) \to 0 \).
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- We construct a \( T \) with these properties such that when inserted in the key inequality contradicts \( \rho(x) \to 0 \).
Since $\rho(x) \to 0$, we can choose a positive non-increasing function $\ell(x) \to 0$ such that $\ell(\log x)/\rho(x) \to \infty$.

**Lemma**

Let $\ell$ be a positive non-increasing function such that $\ell(x) = o(1)$. Then, there is a positive function $L$ such that

$$\ell(x) \ll L(x) = o(1)$$

and an angle $\pi/2 < \theta < \pi$ such that $\mathcal{L}\{L; s\} = \int_0^\infty L(x)e^{-sx}dx$ has analytic continuation to the sector $-\theta < \arg s < \theta$

We choose $L$ as in this lemma. If we manage to show

$$L(\log x) \ll \rho(x),$$

this contradicts $\ell(\log x)/\rho(x) \to \infty$ and hence one must have $\rho(x) \not\to 0$. 

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Wiener-Ikehara theorem
We now consider

\[ T_b(x) := \int_1^x L(\log u) \cos(b \log u) \, du. \]

Its Mellin transform

\[ G_{T_b}(s) = \frac{1}{2s} \left( \mathcal{L}\{L; s - 1 + ib\} + \mathcal{L}\{L; s - 1 - ib\} \right). \]

is analytic in \( \{s : \Re s \geq \alpha, |\Im s| \leq N\} \) for sufficiently large \( b \).

We have the right to apply the key inequality to \( T_b \)

\[
\sup_{x \geq 1} \left| \frac{T_b(x)}{x\rho(x)} \right| \leq C \left( \esssup_{x \geq 1} \left| T'_b(x) \right| + \sup_{\Re s \geq \alpha, |\Im s| \leq N} \left| G_{T_b}(s) \right| \right)
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Further manipulations of this inequality and studying some asymptotics for \( T_b \) lead to

\[ L(\log x) \ll \rho(x). \]
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