Tauberian class estimates for wavelet and non-wavelet transforms of vector-valued distributions

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Transform Methods and Special Functions – TMSF' 2011
6th International Conference, Sofia, Bulgaria, October 21, 2011

In this talk we study vector-valued distributions in terms of integral transforms

$$M_{\varphi}\mathbf{f}(x,y) = (\mathbf{f} * \varphi_y)(x), \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}_+, \tag{1}$$

where $\varphi_y(t) = y^{-n}\varphi(t/y)$. We call such transforms regularizing transforms.

Two important cases can be distinguished:

1. The wavelet case: $\int_{\mathbb{R}^n} \varphi(t)\mathrm{d}t = 0$.
2. The non-wavelet case: $\int_{\mathbb{R}^n} \varphi(t)\mathrm{d}t \neq 0$.

Our aim is:

- To present several precise characterizations of the spaces of distributions with values in Banach spaces in terms of norm size estimates for (1).

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General Notation

- E always denotes a fixed Banach space with norm $\| \cdot \|_E$.
- X stands for a (Hausdorff) locally convex topological vector space.
- $\mathcal{S}'(\mathbb{R}^n, X) = L_b(\mathcal{S}(\mathbb{R}^n), X)$, the space of X-valued tempered distributions.
- $\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$, the upper half-space.
- $\hat{\varphi}$ denotes the Fourier transform.

Statement of the problem

Suppose that **f** takes a priori values in the "broad" space X, i.e.,

- $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, X)$.

Suppose that the "narrower" space

- E is continuously embedded in X.

If we know that **f** takes values in E, $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$, then (for some k, l, C):

$$\|M_{\varphi}\mathbf{f}(x,y)\|_{E} \leq C\frac{(1+y)^{k}(1+|x|)^{l}}{y^{k}}, \quad (x,y) \in \mathbb{H}^{n+1}. \qquad (2)$$

We call (2) a (Tauberian) class estimate.

Converse problem: Up to what extend does the class estimate (2) allow one to conclude that **f** actually takes values in E? The problem has a Tauberian character.

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Motivation

The stated problem was first raised and studied by Drozhzhinov and Zav'yalov. It gives a general setting to attack problems such as:

1. Classical Hardy-Littlewood-Karamata type Tauberian theorems for various integral transforms (e.g., the Laplace transform).
2. Stabilization in time for certain Cauchy problems (e.g., for the heat equation).
3. Norm estimates for solutions to certain PDE (e.g., the Schrödinger equation)
4. Wavelet characterizations of important Banach spaces of functions and distributions (e.g., Besov type spaces).
5. Pointwise and (micro-)local analysis.

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Local class estimates

We said that $M_\varphi \mathbf{f}$ satisfies a local class estimate if:

1. $M_\varphi \mathbf{f}(x,y)$ takes values in E for almost all $(x,y) \in \mathbb{R}^n \times (0,1]$ and is measurable as an E-valued function on $\mathbb{R}^n \times (0,1]$, and,
2. (the local class estimate):

$$\|M_\varphi \mathbf{f}(x,y)\|_E \leq C \frac{(1+|x|)^l}{y^k}, \quad \text{for almost all } (x,y) \in \mathbb{R}^n \times (0,1].$$

for some $k, l \in \mathbb{N}$ and $C > 0$.

Furthermore, we assume from now on that:

- The Banach space E is continuously embedded in the locally convex space X.

Non-degenerate test functions

Naturally, not all kernels φ will be well-suited to our problem.

The good ones are:

Definition

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. It is said to be degenerate if there is a ray through the origin along which $\hat{\varphi}$ identically vanishes. In contrary case, the test function it is said to be non-degenerate.

Our Tauberian kernels are the non-degenerate test functions.

- In Wiener Tauberian theory the Tauberian kernels are those φ such that $\hat{\varphi}$ do not vanish at any point.
- In our theory the Tauberian kernels will be those φ such that $\hat{\varphi}$ do not identically vanish on any ray through the origin.

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The non-wavelet case

For the non-wavelet case, we always obtain a full characterization of $\mathcal{S}'(\mathbb{R}^n, E)$.

Theorem

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, X)$ and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \varphi(t)\mathrm{d}t \neq 0$. Then,

$\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ if and only if $M_\varphi \mathbf{f}$ satisfies a local class estimate

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$\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ *if and only if* $M_\varphi \mathbf{f}$ *satisfies a local class estimate*

The wavelet case

The analysis of the wavelet case is more complicated.

- We only obtain characterizations of $S'(\mathbb{R}^n, E)$ up to a correction term that is totally controlled by the wavelet.
- From now on, we assume that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is a non-degenerate wavelet, namely,
  $\int_{\mathbb{R}^n} \varphi(t)\mathrm{d}t = 0$ and φ is non-degenerate.

Definition

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be non-degenerate. Given $\omega \in \mathbb{S}^{n-1}$, we consider $\hat{\varphi}_\omega(r) := \hat{\varphi}(r\omega)$ as a function of one variable r. We define its index of non-degenerateness as

$$\tau = \inf \left\{ r \in \mathbb{R}_+ : \operatorname{supp} \hat{\varphi}_\omega \cap [0, r] \neq \emptyset, \forall \omega \in \mathbb{S}^{n-1} \right\}.$$

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Wavelet case
Local class estimates

Theorem

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, X)$ *and let* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *be a non-degenerate wavelet with index* τ.

Assume that $M_\varphi \mathbf{f}$ *satisfies a local class estimate.*

Then: for every $r > \tau$, *there is an* X*-valued entire function* $\mathbf{G}$ *such that*

$$\mathbf{f} - \mathbf{G} \in \mathcal{S}'(\mathbb{R}^n, E),$$

where $\operatorname{supp} \hat{\mathbf{G}} \subset \{t \in \mathbb{R}^n : \ |t| < r\}$.

Strongly non-degenerate wavelets

It is still possible to strengthen the previous result, but one should use the following kind of wavelets:

Definition

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a wavelet. We call φ strongly non-degenerate if there exist constants $N \in \mathbb{N}$, $r > 0$, and $C > 0$ such that

$$C\left|u\right|^N \leq \left|\hat{\varphi}(u)\right|, \quad \text{for all } \left|u\right| \leq r.$$

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Theorem

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, X)$ *and let* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *be a strongly non-degenerate wavelet. Then, the following two conditions are equivalent:*

- $M_\varphi \mathbf{f}$ *satisfies a local class estimate.*
- *There is an* X*-valued entire function* $\mathbf{G}$ *such that*

$$\mathbf{f} - \mathbf{G} \in \mathcal{S}'(\mathbb{R}^n, E) \quad \textit{and} \quad \operatorname{supp} \hat{\mathbf{G}} \subseteq \{0\} .$$

Corollary

If X *is a normed space, the function* $\mathbf{G} = \mathbf{P}$ *is indeed a polynomial with coefficients in* X.

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Comments on the (Tauberian) theorems

The discussed theorems improve several earlier results of Drozhzhinov and Zav'ylov.

Main improvements:

- Enlargement of the Tauberian kernels. Actually, our class of non-degenerate kernels is the optimal one.
- Our results are valid for general locally convex spaces X (Drozhzhinov and Zav'ylov considered normed spaces).

References

For further results see our preprint (joint work with S. Pilipović):

- Multidimensional Tauberian theorems for wavelets and non-wavelet transforms, preprint (arXiv:1012.5090v2).

See also:

- Y. N. Drozhzhinov, B. I. Zav'yalov, Tauberian theorems for generalized functions with values in Banach spaces, Izv. Math. 66 (2002), 701–769.
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