Recent developments on complex Tauberian theorems for Laplace transforms

Jasson Vindas
jvindas@cage.UGent.be

Department of Mathematics
Ghent University

International Conference on Generalized Functions – GF2016
Dubrovnik, Croatia, September 8, 2016

Complex Tauberian theorems have been strikingly useful tools in diverse areas such as:

- Analytic number theory and analytic combinatorics.
- Spectral theory for (pseudo-)differential operators.
- Last three decades: operator theory and semigroups.

We will discuss some recent developments on complex Tauberians for Laplace transforms and power series. We will be concerned with two groups of statements:

- Wiener-Ikehara theorems.
- Ingham-Fatou-Riesz theorems.

Main questions:

1. Relax boundary requirements to a minimum.
2. Mild Tauberian hypotheses (one-sided conditions).

This talk is based on collaborative work with G. Debruyne.

Complex Tauberian theorems have been strikingly useful tools in diverse areas such as:

- Analytic number theory and analytic combinatorics.
- Spectral theory for (pseudo-)differential operators.
- Last three decades: operator theory and semigroups.

We will discuss some recent developments on complex Tauberians for Laplace transforms and power series. We will be concerned with two groups of statements:

- Wiener-Ikehara theorems.
- Ingham-Fatou-Riesz theorems.

Main questions:

1. Relax boundary requirements to a minimum.
2. Mild Tauberian hypotheses (one-sided conditions).

This talk is based on collaborative work with G. Debruyne.

Complex Tauberian theorems have been strikingly useful tools in diverse areas such as:

- Analytic number theory and analytic combinatorics.
- Spectral theory for (pseudo-)differential operators.
- Last three decades: operator theory and semigroups.

We will discuss some recent developments on complex Tauberians for Laplace transforms and power series. We will be concerned with two groups of statements:

- Wiener-Ikehara theorems.
- Ingham-Fatou-Riesz theorems.

Main questions:

1. Relax boundary requirements to a minimum.
2. Mild Tauberian hypotheses (one-sided conditions).

This talk is based on collaborative work with G. Debruyne.

Complex Tauberian theorems have been strikingly useful tools in diverse areas such as:

- Analytic number theory and analytic combinatorics.
- Spectral theory for (pseudo-)differential operators.
- Last three decades: operator theory and semigroups.

We will discuss some recent developments on complex Tauberians for Laplace transforms and power series. We will be concerned with two groups of statements:

- Wiener-Ikehara theorems.
- Ingham-Fatou-Riesz theorems.

Main questions:

1. Relax boundary requirements to a minimum.
2. Mild Tauberian hypotheses (one-sided conditions).

This talk is based on collaborative work with G. Debruyne.

The classical Wiener-Ikehara theorem

Theorem (Wiener-Ikehara, Laplace-Stieltjes transforms)

Let S be a non-decreasing function (Tauberian hypothesis) such that $\mathcal{L}\{\mathrm{d}S; z\} = \int_{0^-}^{\infty} e^{-zt}\mathrm{d}S(t)$ converges for $\Re e\, z > 1$. If

$$\mathcal{L}\{\mathrm{d}S; z\} - \frac{A}{z-1}$$

has analytic continuation through $\Re e\, z = 1$, then $S(x) \sim Ae^x$.

Theorem (Wiener-Ikehara, version for Dirichlet series)

Let $a_n \geq 0$ and $\lambda_n \nearrow \infty$. Suppose $\sum_{n=1}^{\infty} a_n \lambda_n^{-z}$ converges for $\Re e\, z > 1$. If

$$\sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^z} - \frac{A}{z-1}$$

has analytic continuation through $\Re e\, z = 1$, then $\sum_{\lambda_n \leq x} a_n \sim Ax$.

The classical Wiener-Ikehara theorem

Theorem (Wiener-Ikehara, Laplace-Stieltjes transforms)

Let S be a non-decreasing function (Tauberian hypothesis) such that $\mathcal{L}\{\mathrm{d}S; z\} = \int_{0^-}^{\infty} e^{-zt}\mathrm{d}S(t)$ converges for $\Re e\, z > 1$. If

$$\mathcal{L}\{\mathrm{d}S; z\} - \frac{A}{z-1}$$

has analytic continuation through $\Re e\, z = 1$, then $S(x) \sim Ae^{x}$.

Theorem (Wiener-Ikehara, version for Dirichlet series)

Let $a_n \geq 0$ and $\lambda_n \nearrow \infty$. Suppose $\sum_{n=1}^{\infty} a_n \lambda_n^{-z}$ converges for $\Re e\, z > 1$. If

$$\sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^{z}} - \frac{A}{z-1}$$

has analytic continuation through $\Re e\, z = 1$, then $\sum_{\lambda_n \leq x} a_n \sim Ax$.

The classical Wiener-Ikehara theorem

Theorem (Wiener-Ikehara, Laplace-Stieltjes transforms)

Let S be a non-decreasing function (Tauberian hypothesis) such that $\mathcal{L}\{\mathrm{d}S; z\} = \int_{0^-}^{\infty} e^{-zt}\mathrm{d}S(t)$ converges for $\Re e\, z > 1$. If

$$\mathcal{L}\{\mathrm{d}S; z\} - \frac{A}{z-1}$$

has analytic continuation through $\Re e\, z = 1$, then $S(x) \sim Ae^{x}$.

Theorem (Wiener-Ikehara, version for Dirichlet series)

Let $a_n \geq 0$ and $\lambda_n \nearrow \infty$. Suppose $\sum_{n=1}^{\infty} a_n \lambda_n^{-z}$ converges for $\Re e\, z > 1$. If

$$\sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^z} - \frac{A}{z-1}$$

has analytic continuation through $\Re e\, z = 1$, then $\sum_{\lambda_n \leq x} a_n \sim Ax$.

From the Wiener-Ikehara theorem to the PNT:

The Prime Number Theorem (PNT) asserts that

$$\pi(x) = \sum_{p \le x} 1 \sim \frac{x}{\log x}$$

- PNT is equivalent to $\psi(x) = \sum_{p^{\alpha} \le x} \log p = \sum_{n \le x} \Lambda(n) \sim x$.
- $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ has analytic continuation to $\Re e\, z > 0$ except for simple pole with residue 1 at $z = 1$.
- Logarithmic differentiation of $\zeta(z) = \prod_p (1 - p^{-z})^{-1}$ leads to

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z} = -\frac{\zeta'(z)}{\zeta(z)}, \quad \Re e\, z > 1.$$

- $(z-1)\zeta(z)$ has no zeros on $\Re e\, z = 1$, so

$$-\frac{d}{dz}(\log((z-1)\zeta(z))) = -\frac{\zeta'(z)}{\zeta(z)} - \frac{1}{z-1}$$

is analytic in a region containing $\Re e\, z \ge 1$. The rest follows from the Wiener-Ikehara theorem.

From the Wiener-Ikehara theorem to the PNT:

The Prime Number Theorem (PNT) asserts that

$$\pi(x) = \sum_{p \leq x} 1 \sim \frac{x}{\log x}$$

- PNT is equivalent to $\psi(x) = \sum_{p^\alpha \leq x} \log p = \sum_{n \leq x} \Lambda(n) \sim x$.
- $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ has analytic continuation to $\Re e\, z > 0$ except for simple pole with residue 1 at $z = 1$.
- Logarithmic differentiation of $\zeta(z) = \prod_p (1 - p^{-z})^{-1}$ leads to

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z} = -\frac{\zeta'(z)}{\zeta(z)}, \quad \Re e\, z > 1.$$

- $(z-1)\zeta(z)$ has no zeros on $\Re e\, z = 1$, so

$$-\frac{d}{dz}(\log((z-1)\zeta(z))) = -\frac{\zeta'(z)}{\zeta(z)} - \frac{1}{z-1}$$

  is analytic in a region containing $\Re e\, z \geq 1$. The rest follows from the Wiener-Ikehara theorem.

From the Wiener-Ikehara theorem to the PNT:

The Prime Number Theorem (PNT) asserts that

$$\pi(x) = \sum_{p \le x} 1 \sim \frac{x}{\log x}$$

- PNT is equivalent to $\psi(x) = \sum_{p^\alpha \le x} \log p = \sum_{n \le x} \Lambda(n) \sim x$.
- $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ has analytic continuation to $\Re e\, z > 0$ except for simple pole with residue 1 at $z = 1$.
- Logarithmic differentiation of $\zeta(z) = \prod_p (1 - p^{-z})^{-1}$ leads to

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z} = -\frac{\zeta'(z)}{\zeta(z)}, \quad \Re e\, z > 1.$$

- $(z-1)\zeta(z)$ has no zeros on $\Re e\, z = 1$, so

$$-\frac{d}{dz}(\log((z-1)\zeta(z))) = -\frac{\zeta'(z)}{\zeta(z)} - \frac{1}{z-1}$$

is analytic in a region containing $\Re e\, z \ge 1$. The rest follows from the Wiener-Ikehara theorem.

From the Wiener-Ikehara theorem to the PNT:

The Prime Number Theorem (PNT) asserts that

$$\pi(x) = \sum_{p \leq x} 1 \sim \frac{x}{\log x}$$

- PNT is equivalent to $\psi(x) = \sum_{p^\alpha \leq x} \log p = \sum_{n \leq x} \Lambda(n) \sim x$.
- $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ has analytic continuation to $\Re e\, z > 0$ except for simple pole with residue 1 at $z = 1$.
- Logarithmic differentiation of $\zeta(z) = \prod_p (1 - p^{-z})^{-1}$ leads to

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z} = -\frac{\zeta'(z)}{\zeta(z)}, \quad \Re e\, z > 1.$$

- $(z-1)\zeta(z)$ has no zeros on $\Re e\, z = 1$, so

$$-\frac{d}{dz}(\log((z-1)\zeta(z))) = -\frac{\zeta'(z)}{\zeta(z)} - \frac{1}{z-1}$$

is analytic in a region containing $\Re e\, z \geq 1$. The rest follows from the Wiener-Ikehara theorem.

From the Wiener-Ikehara theorem to the PNT:

The Prime Number Theorem (PNT) asserts that

$$\pi(x) = \sum_{p \leq x} 1 \sim \frac{x}{\log x}$$

- PNT is equivalent to $\psi(x) = \sum_{p^{\alpha} \leq x} \log p = \sum_{n \leq x} \Lambda(n) \sim x$.
- $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ has analytic continuation to $\Re e\, z > 0$ except for simple pole with residue 1 at $z = 1$.
- Logarithmic differentiation of $\zeta(z) = \prod_p (1 - p^{-z})^{-1}$ leads to

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z} = -\frac{\zeta'(z)}{\zeta(z)}, \quad \Re e\, z > 1.$$

- $(z-1)\zeta(z)$ has no zeros on $\Re e\, z = 1$, so

$$-\frac{d}{dz}(\log((z-1)\zeta(z))) = -\frac{\zeta'(z)}{\zeta(z)} - \frac{1}{z-1}$$

is analytic in a region containing $\Re e\, z \geq 1$. The rest follows from the Wiener-Ikehara theorem.

Remarks on the Wiener-Ikehara theorem

- Another typical application: Weyl type spectral asymptotics for (pseudo-)differential operators.
- Historically, the Wiener-Ikehara theorem improved a Tauberian theorem of Landau (1908) by eliminating the unnecessary hypothesis $G(z) = O(|z|^N)$ on

$$G(z) = \mathcal{L}\{\mathrm{d}S; z\} - \frac{A}{z-1}$$

- The hypothesis $G(z)$ has analytic continuation to $\Re e\, z = 1$ can be significantly relaxed to "good boundary behavior":
  1. G(z) has continuous extension to $\Re e\, z = 1$.
  2. L^1_{loc}-boundary behavior: $\lim_{x \to 1^+} G(x + iy) \in L^1(I)$ for every finite interval I.
  3. Local pseudofunction boundary behavior (Korevaar, 2005). To be explained later ...
  4. Local pseudofunction boundary behavior except on a small set where additional conditions hold (Debruyne-V., 2016).

Remarks on the Wiener-Ikehara theorem

- Another typical application: Weyl type spectral asymptotics for (pseudo-)differential operators.
- Historically, the Wiener-Ikehara theorem improved a Tauberian theorem of Landau (1908) by eliminating the unnecessary hypothesis $G(z) = O(|z|^N)$ on

$$G(z) = \mathcal{L}\{\mathrm{d}S; z\} - \frac{A}{z-1}$$

- The hypothesis $G(z)$ has analytic continuation to $\Re e\, z = 1$ can be significantly relaxed to "good boundary behavior":
  1. G(z) has continuous extension to $\Re e\, z = 1$.
  2. L^1_{loc}-boundary behavior: $\lim_{x\to 1^+} G(x+iy) \in L^1(I)$ for every finite interval I.
  3. Local pseudofunction boundary behavior (Korevaar, 2005). To be explained later ...
  4. Local pseudofunction boundary behavior except on a small set where additional conditions hold (Debruyne-V., 2016).

Remarks on the Wiener-Ikehara theorem

- Another typical application: Weyl type spectral asymptotics for (pseudo-)differential operators.
- Historically, the Wiener-Ikehara theorem improved a Tauberian theorem of Landau (1908) by eliminating the unnecessary hypothesis $G(z) = O(|z|^N)$ on

$$G(z) = \mathcal{L}\{\mathrm{d}S; z\} - \frac{A}{z-1}$$

- The hypothesis $G(z)$ has analytic continuation to $\Re e\, z = 1$ can be significantly relaxed to "good boundary behavior":
  1. G(z) has continuous extension to $\Re e\, z = 1$.
  2. L^1_{loc}-boundary behavior: $\lim_{x \to 1^+} G(x + iy) \in L^1(I)$ for every finite interval I.
  3. Local pseudofunction boundary behavior (Korevaar, 2005). To be explained later ...
  4. Local pseudofunction boundary behavior except on a small set where additional conditions hold (Debruyne-V., 2016).

Remarks on the Wiener-Ikehara theorem

- Another typical application: Weyl type spectral asymptotics for (pseudo-)differential operators.
- Historically, the Wiener-Ikehara theorem improved a Tauberian theorem of Landau (1908) by eliminating the unnecessary hypothesis $G(z) = O(|z|^N)$ on

$$G(z) = \mathcal{L}\{\mathrm{d}S; z\} - \frac{A}{z-1}$$

- The hypothesis $G(z)$ has analytic continuation to $\Re e\, z = 1$ can be significantly relaxed to “good boundary behavior":
  1. G(z) has continuous extension to $\Re e\, z = 1$.
  2. L^1_{loc}-boundary behavior: $\lim_{x \to 1^+} G(x + iy) \in L^1(I)$ for every finite interval I.
  3. Local pseudofunction boundary behavior (Korevaar, 2005). To be explained later ...
  4. Local pseudofunction boundary behavior except on a small set where additional conditions hold (Debruyne-V., 2016).

Remarks on the Wiener-Ikehara theorem

- Another typical application: Weyl type spectral asymptotics for (pseudo-)differential operators.
- Historically, the Wiener-Ikehara theorem improved a Tauberian theorem of Landau (1908) by eliminating the unnecessary hypothesis $G(z) = O(|z|^N)$ on

$$G(z) = \mathcal{L}\{\mathrm{d}S; z\} - \frac{A}{z-1}$$

- The hypothesis $G(z)$ has analytic continuation to $\Re e\, z = 1$ can be significantly relaxed to "good boundary behavior":
  1. G(z) has continuous extension to $\Re e\, z = 1$.
  2. L^1_{loc}-boundary behavior: $\lim_{x \to 1^+} G(x + iy) \in L^1(I)$ for every finite interval I.
  3. Local pseudofunction boundary behavior (Korevaar, 2005). To be explained later ...
  4. Local pseudofunction boundary behavior except on a small set where additional conditions hold (Debruyne-V., 2016).

Remarks on the Wiener-Ikehara theorem

- Another typical application: Weyl type spectral asymptotics for (pseudo-)differential operators.
- Historically, the Wiener-Ikehara theorem improved a Tauberian theorem of Landau (1908) by eliminating the unnecessary hypothesis $G(z) = O(|z|^N)$ on

$$G(z) = \mathcal{L}\{\mathrm{d}S; z\} - \frac{A}{z-1}$$

- The hypothesis $G(z)$ has analytic continuation to $\Re e\, z = 1$ can be significantly relaxed to "good boundary behavior":
  1. G(z) has continuous extension to $\Re e\, z = 1$.
  2. L^1_{loc}-boundary behavior: $\lim_{x\to 1^+} G(x+iy) \in L^1(I)$ for every finite interval I.
  3. Local pseudofunction boundary behavior (Korevaar, 2005). To be explained later ...
  4. Local pseudofunction boundary behavior except on a small set where additional conditions hold (Debruyne-V., 2016).

Remarks on the Wiener-Ikehara theorem

- Another typical application: Weyl type spectral asymptotics for (pseudo-)differential operators.
- Historically, the Wiener-Ikehara theorem improved a Tauberian theorem of Landau (1908) by eliminating the unnecessary hypothesis $G(z) = O(|z|^N)$ on

$$G(z) = \mathcal{L}\{\mathrm{d}S; z\} - \frac{A}{z-1}$$

- The hypothesis $G(z)$ has analytic continuation to $\Re e\, z = 1$ can be significantly relaxed to "good boundary behavior":
  1. G(z) has continuous extension to $\Re e\, z = 1$.
  2. L^1_{loc}-boundary behavior: $\lim_{x \to 1^+} G(x + iy) \in L^1(I)$ for every finite interval I.
  3. Local pseudofunction boundary behavior (Korevaar, 2005). To be explained later ...
  4. Local pseudofunction boundary behavior except on a small set where additional conditions hold (Debruyne-V., 2016).

The Fatou-Riesz theorem

In his very influential 1906 paper

Séries trigonométriques et séries de Taylor,

Fatou proved the following theorem on analytic continuation of power series.

Theorem (Fatou-Riesz theorem)

Suppose that $F(z) = \sum_{n=0}^{\infty} c_n z^n$ *converges for* $|z| < 1$ *and* $c_n = o(1)$ *(this is the Tauberian condition). If* $F(z)$ *has analytic continuation to a neigborhood of* $z = 1$*, then* $\sum_{n=0}^{\infty} c_n$*, converges and*

$$\sum_{n=0}^{\infty} c_n = F(1).$$

Marcel Riesz gave three proofs of this theorem (1909, 1911, 1916), so his name is usually associated to this result.

The Fatou-Riesz theorem

In his very influential 1906 paper

Séries trigonométriques et séries de Taylor,

Fatou proved the following theorem on analytic continuation of power series.

Theorem (Fatou-Riesz theorem)

Suppose that $F(z) = \sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < 1$ and $c_n = o(1)$ (this is the Tauberian condition). If $F(z)$ has analytic continuation to a neigborhood of $z = 1$, then $\sum_{n=0}^{\infty} c_n$, converges and

$$\sum_{n=0}^{\infty} c_n = F(1).$$

Marcel Riesz gave three proofs of this theorem (1909, 1911, 1916), so his name is usually associated to this result.

Ingham theorem for Laplace transforms

In 1935 Ingham obtained a Fatou-Riesz type Tauberian theorem for Laplace transforms. Essentially the same theorem was shown by Karamata in 1936. The result makes use of *slow decrease*.

A function τ is called slowly decreasing if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\liminf_{x\to\infty} \inf_{h\in[0,\delta]} (\tau(x+h) - \tau(x)) > -\varepsilon.$$

that is, $\tau(x+h) - \tau(x) > -\varepsilon$ for $x > X_\varepsilon$ and $0 \le h < \delta_\varepsilon$.

Theorem (Ingham)

Let $\tau \in L^1_{loc}(\mathbb{R})$ be slowly decreasing (Tauberian hypothesis), vanish on $(-\infty, 0)$, and have convergent Laplace transform $\mathcal{L}\{\tau; z\} = \int_0^\infty \tau(t)e^{-zt}\mathrm{d}t$ for $\Re e\, z > 0$. Suppose that there is a constant b such that

$$\mathcal{L}\{\tau; z\} - \frac{b}{z}$$

has L^1_{loc}-boundary behavior on $\Re e\, z = 0$, then $\lim_{x\to\infty} \tau(x) = b$.

Ingham theorem for Laplace transforms

In 1935 Ingham obtained a Fatou-Riesz type Tauberian theorem for Laplace transforms. Essentially the same theorem was shown by Karamata in 1936. The result makes use of *slow decrease*.

A function τ is called slowly decreasing if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\liminf_{x\to\infty} \inf_{h\in[0,\delta]} (\tau(x+h) - \tau(x)) > -\varepsilon.$$

that is, $\tau(x+h) - \tau(x) > -\varepsilon$ for $x > X_\varepsilon$ and $0 \le h < \delta_\varepsilon$.

Theorem (Ingham)

Let $\tau \in L^1_{loc}(\mathbb{R})$ be slowly decreasing (Tauberian hypothesis), vanish on $(-\infty, 0)$, and have convergent Laplace transform $\mathcal{L}\{\tau; z\} = \int_0^\infty \tau(t)e^{-zt}\mathrm{d}t$ for $\Re e\, z > 0$. Suppose that there is a constant b such that

$$\mathcal{L}\{\tau; z\} - \frac{b}{z}$$

has L^1_{loc}-boundary behavior on $\Re e\, z = 0$, then $\lim_{x\to\infty} \tau(x) = b$.

Ingham theorem for Laplace transforms

In 1935 Ingham obtained a Fatou-Riesz type Tauberian theorem for Laplace transforms. Essentially the same theorem was shown by Karamata in 1936. The result makes use of *slow decrease*.

A function τ is called slowly decreasing if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\liminf_{x\to\infty} \inf_{h\in[0,\delta]} (\tau(x+h) - \tau(x)) > -\varepsilon.$$

that is, $\tau(x+h) - \tau(x) > -\varepsilon$ for $x > X_\varepsilon$ and $0 \le h < \delta_\varepsilon$.

Theorem (Ingham)

Let $\tau \in L^1_{loc}(\mathbb{R})$ be slowly decreasing (Tauberian hypothesis), vanish on $(-\infty, 0)$, and have convergent Laplace transform $\mathcal{L}\{\tau; z\} = \int_0^\infty \tau(t)e^{-zt}\mathrm{d}t$ for $\Re e\, z > 0$. Suppose that there is a constant b such that

$$\mathcal{L}\{\tau; z\} - \frac{b}{z}$$

has L^1_{loc}-boundary behavior on $\Re e\, z = 0$, then $\lim_{x\to\infty} \tau(x) = b$.

Developments in the 1980s: Newman's contour integration method

In 1980 Newman gave a simple contour integration proof of the next Tauberian theorem.

Theorem

Let $a_n = O(1)$ (Tauberian hypothesis). If $F(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z}$ has analytic continuation beyond $\Re e\, z = 1$, then

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = F(1).$$

This Tauberian theorem is contained in Ingham's theorem; however, Newman's contour integration proof is shorter and simpler.

Developments in the 1980s: Newman's contour integration method

In 1980 Newman gave a simple contour integration proof of the next Tauberian theorem.

Theorem

Let $a_n = O(1)$ (Tauberian hypothesis). If $F(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z}$ has analytic continuation beyond $\Re e\, z = 1$, then

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = F(1).$$

This Tauberian theorem is contained in Ingham's theorem; however, Newman's contour integration proof is shorter and simpler.

Newman's short way to the PNT

Newman's Tauberian theorem from above provides a relatively simple way to prove the PNT.

- One works here with the Möbius

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n \text{ has } r \text{ distinct prime factors}, \\ 0 & \textit{otherwise}. \end{cases}$$

- Property: μ is the Dirichlet convolution inverse of 1. So,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} = \frac{1}{\zeta(z)} \quad (\zeta \text{ is the Riemann zeta function})$$

- Applying the previous theorem, $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = \frac{1}{\zeta(0)} = 0.$
- The latter relation was shown to imply the PNT by Landau in 1913 via elementary (real-variable) methods.

Newman's short way to the PNT

Newman's Tauberian theorem from above provides a relatively simple way to prove the PNT.

- One works here with the Möbius

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n \text{ has } r \text{ distinct prime factors}, \\ 0 & \textit{otherwise}. \end{cases}$$

- Property: μ is the Dirichlet convolution inverse of 1. So,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} = \frac{1}{\zeta(z)} \quad (\zeta \text{ is the Riemann zeta function})$$

- Applying the previous theorem, $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = \frac{1}{\zeta(0)} = 0.$
- The latter relation was shown to imply the PNT by Landau in 1913 via elementary (real-variable) methods.

Newman's short way to the PNT

Newman's Tauberian theorem from above provides a relatively simple way to prove the PNT.

- One works here with the Möbius

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n \text{ has } r \text{ distinct prime factors,} \\ 0 & \textit{otherwise}. \end{cases}$$

- Property: μ is the Dirichlet convolution inverse of 1. So,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} = \frac{1}{\zeta(z)} \quad (\zeta \text{ is the Riemann zeta function})$$

- Applying the previous theorem, $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = \frac{1}{\zeta(0)} = 0.$
- The latter relation was shown to imply the PNT by Landau in 1913 via elementary (real-variable) methods.

Tauberians motivated by applications in semigroups

Newman's contour integration method was adapted to a variety of Tauberian problems in numerous articles.

Its importance was recognized by the semigroup community. Here is a sample (extending a result of Korevaar and Zagier):

Theorem (Arendt and Batty, 1988)

Let $\rho \in L^{\infty}(\mathbb{R})$ (Tauberian hypothesis) vanish on $(-\infty, 0)$. Suppose that $\mathcal{L}\{\rho; z\}$ has analytic continuation at every point of the complement of iE where $E \subset \mathbb{R}$ is a closed null set. If $0 \notin iE$ and

$$\sup_{t\in E} \sup_{x>0} \left| \int_0^x e^{-itu} \rho(u) \mathrm{d}u \right| < \infty,$$

then the (improper) integral of ρ converges to $b = \mathcal{L}\{\rho; 0\}$, that is,

$$\int_0^{\infty} \rho(t) \mathrm{d}t = b.$$

Tauberians motivated by applications in semigroups

Newman's contour integration method was adapted to a variety of Tauberian problems in numerous articles.

Its importance was recognized by the semigroup community. Here is a sample (extending a result of Korevaar and Zagier):

Theorem (Arendt and Batty, 1988)

Let $\rho \in L^{\infty}(\mathbb{R})$ (Tauberian hypothesis) vanish on $(-\infty, 0)$. Suppose that $\mathcal{L}\{\rho; z\}$ has analytic continuation at every point of the complement of iE where $E \subset \mathbb{R}$ is a closed null set. If $0 \notin iE$ and

$$\sup_{t\in E}\sup_{x>0}\left|\int_0^x e^{-itu}\rho(u)\mathrm{d}u\right| < \infty,$$

then the (improper) integral of ρ converges to $b = \mathcal{L}\{\rho; 0\}$, that is,

$$\int_0^{\infty}\rho(t)\mathrm{d}t = b.$$

If $E = \emptyset$, the result is due to Korevaar and Zagier (independently), who also obtained it via Newman's contour integration technique.

In this case, the result is contained in Ingham's Fatou-Riesz type theorem:

- Set $\tau(x) = \int_0^x \rho(u)\mathrm{d}u \Rightarrow \mathcal{L}\{\tau; z\} = \dfrac{\mathcal{L}\{\rho; z\}}{z}$.
- $\mathcal{L}\{\rho; z\}$ has analytic continuation beyond $\Re e\, z = 0$ if and only if

$$\mathcal{L}\{\tau; z\} - \frac{b}{z} = \frac{\mathcal{L}\{\rho; z\} - b}{z}$$

does with $b = \mathcal{L}\{\rho; 0\}$.

If $E = \emptyset$, the result is due to Korevaar and Zagier (independently), who also obtained it via Newman's contour integration technique.

In this case, the result is contained in Ingham's Fatou-Riesz type theorem:

- Set $\tau(x) = \int_0^x \rho(u)\mathrm{d}u \Rightarrow \mathcal{L}\{\tau; z\} = \dfrac{\mathcal{L}\{\rho; z\}}{z}$.
- $\mathcal{L}\{\rho; z\}$ has analytic continuation beyond $\Re e\, z = 0$ if and only if

$$\mathcal{L}\{\tau; z\} - \frac{b}{z} = \frac{\mathcal{L}\{\rho; z\} - b}{z}$$

does with $b = \mathcal{L}\{\rho; 0\}$.

If $E = \emptyset$, the result is due to Korevaar and Zagier (independently), who also obtained it via Newman's contour integration technique.

In this case, the result is contained in Ingham's Fatou-Riesz type theorem:

- Set $\tau(x) = \int_0^x \rho(u)\mathrm{d}u \Rightarrow \mathcal{L}\{\tau; z\} = \dfrac{\mathcal{L}\{\rho; z\}}{z}$.
- $\mathcal{L}\{\rho; z\}$ has analytic continuation beyond $\Re e\, z = 0$ if and only if
$$\mathcal{L}\{\tau; z\} - \frac{b}{z} = \frac{\mathcal{L}\{\rho; z\} - b}{z}$$
does with $b = \mathcal{L}\{\rho; 0\}$.

The Arendt-Batty Tauberian theorem readily extends to functions with values on a Banach space. Here is a sample application of the vector-valued version:

Theorem (Arendt and Batty)

Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a reflexive Banach space X. Denote the spectrum of its infinitesimal generator A as $\sigma(A)$. If $\sigma(A)\cap i\mathbb{R}$ is countable and no eigenvalue of A lies on the imaginary axis, then

$$\lim_{t\to\infty} T(t)x = 0, \quad \forall x \in X.$$

In recent times, Tauberian methods have been revisited to study rates of converge that can be a applied to PDE, e.g. decay estimates for damped wave equations.

The Arendt-Batty Tauberian theorem readily extends to functions with values on a Banach space. Here is a sample application of the vector-valued version:

Theorem (Arendt and Batty)

Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a reflexive Banach space X. Denote the spectrum of its infinitesimal generator A as $\sigma(A)$. If $\sigma(A)\cap i\mathbb{R}$ is countable and no eigenvalue of A lies on the imaginary axis, then

$$\lim_{t\to\infty} T(t)x = 0, \quad \forall x \in X.$$

In recent times, Tauberian methods have been revisited to study rates of converge that can be a applied to PDE, e.g. decay estimates for damped wave equations.

Input from operator theory

In 1986 Katznelson and Tzafriri proved the next interesting theorem for power series. Denote as $\mathbb{D}$ the unit disc in the complex plane.

Theorem (Katznelson and Tzafriri)

Suppose that $F(z) = \sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < 1$ and $S_n = \sum_{k=0}^{n} c_k = O(1)$ (Tauberian condition). If $F(z)$ has analytic continuation to every point $\partial\mathbb{D} \setminus \{1\}$, then $c_n = o(1)$.

Katznelson and Tzafriri obtained their theorem under weaker assumptions than analytic continuation, namely, in terms of local pseudofunction behavior, initiating so the distributional approach in complex Tauberian theory.

Input from operator theory

In 1986 Katznelson and Tzafriri proved the next interesting theorem for power series. Denote as $\mathbb{D}$ the unit disc in the complex plane.

Theorem (Katznelson and Tzafriri)

Suppose that $F(z) = \sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < 1$ and $S_n = \sum_{k=0}^{n} c_k = O(1)$ (Tauberian condition). If $F(z)$ has analytic continuation to every point $\partial\mathbb{D} \setminus \{1\}$, then $c_n = o(1)$.

Katznelson and Tzafriri obtained their theorem under weaker assumptions than analytic continuation, namely, in terms of local pseudofunction behavior, initiating so the distributional approach in complex Tauberian theory.

Application in operator theory

The above theorem has also a Banach space valued counterpart, which Katznelson and Tzafriri used to prove:

Theorem (Katznelson and Tzafriri, 1986)

Let T be a power-bounded operator on a Banach space ($\sup_{n\in\mathbb{N}}\|T^n\|<\infty$). Then,

$$\lim_{n\to\infty}\|T^{n+1}-T^n\|=0$$

if and only if $\sigma(T)\cap\partial\mathbb{D}\subseteq\{1\}$.

Proof: The contraposition of the direct implication follows by standard functional calculus. For the converse, if $\lambda I-T$ is invertible for all $|\lambda|\geq 1$, $\lambda\neq 1$, then $g(z)=\sum_{n=0}^{\infty}T^n z^n$ is analytic on $\partial\mathbb{D}\setminus\{1\}$, the same is true for

$$F(z)=(I-z)g(z)=\sum_{n=0}^{\infty}(T^n-T^{n+1})z^n\Rightarrow\|T^{n+1}-T^n\|\to 0.$$

Application in operator theory

The above theorem has also a Banach space valued counterpart, which Katznelson and Tzafriri used to prove:

Theorem (Katznelson and Tzafriri, 1986)

Let T be a power-bounded operator on a Banach space ($\sup_{n\in\mathbb{N}} \|T^n\| < \infty$). Then,

$$\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0$$

if and only if $\sigma(T) \cap \partial\mathbb{D} \subseteq \{1\}$.

Proof: The contraposition of the direct implication follows by standard functional calculus. For the converse, if $\lambda I - T$ is invertible for all $|\lambda| \geq 1$, $\lambda \neq 1$, then $g(z) = \sum_{n=0}^{\infty} T^n z^n$ is analytic on $\partial\mathbb{D} \setminus \{1\}$, the same is true for

$$F(z) = (I - z)g(z) = \sum_{n=0}^{\infty}(T^n - T^{n+1})z^n \Rightarrow \|T^{n+1} - T^n\| \to 0.$$

Application in operator theory

The above theorem has also a Banach space valued counterpart, which Katznelson and Tzafriri used to prove:

Theorem (Katznelson and Tzafriri, 1986)

Let T be a power-bounded operator on a Banach space ($\sup_{n\in\mathbb{N}} \|T^n\| < \infty$). Then,

$$\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0$$

if and only if $\sigma(T) \cap \partial\mathbb{D} \subseteq \{1\}$.

Proof: The contraposition of the direct implication follows by standard functional calculus. For the converse, if $\lambda I - T$ is invertible for all $|\lambda| \geq 1$, $\lambda \neq 1$, then $g(z) = \sum_{n=0}^{\infty} T^n z^n$ is analytic on $\partial\mathbb{D} \setminus \{1\}$, the same is true for

$$F(z) = (I - z)g(z) = \sum_{n=0}^{\infty} (T^n - T^{n+1}) z^n \Rightarrow \|T^{n+1} - T^n\| \to 0.$$

Pseudofunctions and pseudomeasures

Pseudofunctions and pseudomeasures are notions that naturally arise in harmonic analysis.

- Pseudomeasures: $PM(\mathbb{R}) = \{g \in \mathcal{S}'(\mathbb{R}) : \widehat{g} \in L^{\infty}(\mathbb{R})\}$
- Pseudofunctions: $PF(\mathbb{R}) = \{g \in PM(\mathbb{R}) : \lim_{x\to\infty} \widehat{g}(x) = 0\}$

Given an open set $U \subseteq \mathbb{R}$, we define the local spaces:

- $PM_{loc}(U) = \{g \in \mathcal{D}'(U) : \varphi g \in PM(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $PF_{loc}(U) = \{g \in \mathcal{D}'(U) : \varphi g \in PF(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $L^1_{loc}(U) \subset PF_{loc}(U)$.
- Every Radon measure on U is a local pseudomeasure.

Let G be analytic on $\Re e\, z > \alpha$ and $U \subset \mathbb{R}$ be open.

We say that G has local pseudofunction boundary behavior on $\alpha + iU$ if it has distributional boundary values there, i.e.

$$\lim_{x\to\alpha^+} G(x+iy) = g(y) \text{ in } \mathcal{D}'(U)$$

and $g \in PF_{loc}(U)$.

Analogously, local pseudomeasure boundary behavior.

Pseudofunctions and pseudomeasures

Pseudofunctions and pseudomeasures are notions that naturally arise in harmonic analysis.

- Pseudomeasures: $PM(\mathbb{R}) = \{g \in \mathcal{S}'(\mathbb{R}) : \widehat{g} \in L^{\infty}(\mathbb{R})\}$
- Pseudofunctions: $PF(\mathbb{R}) = \{g \in PM(\mathbb{R}) : \lim_{x\to\infty} \widehat{g}(x) = 0\}$

Given an open set $U \subseteq \mathbb{R}$, we define the local spaces:

- $PM_{loc}(U) = \{g \in \mathcal{D}'(U) : \varphi g \in PM(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $PF_{loc}(U) = \{g \in \mathcal{D}'(U) : \varphi g \in PF(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $L^1_{loc}(U) \subset PF_{loc}(U)$.
- Every Radon measure on U is a local pseudomeasure.

Let G be analytic on $\Re e\, z > \alpha$ and $U \subset \mathbb{R}$ be open.

We say that G has local pseudofunction boundary behavior on $\alpha + iU$ if it has distributional boundary values there, i.e.

$$\lim_{x\to\alpha^+} G(x+iy) = g(y) \text{ in } \mathcal{D}'(U)$$

and $g \in PF_{loc}(U)$.

Analogously, local pseudomeasure boundary behavior.

Pseudofunctions and pseudomeasures

Pseudofunctions and pseudomeasures are notions that naturally arise in harmonic analysis.

- Pseudomeasures: $PM(\mathbb{R}) = \{g \in \mathcal{S}'(\mathbb{R}) : \widehat{g} \in L^{\infty}(\mathbb{R})\}$
- Pseudofunctions: $PF(\mathbb{R}) = \{g \in PM(\mathbb{R}) : \lim_{x \to \infty} \widehat{g}(x) = 0\}$

Given an open set $U \subseteq \mathbb{R}$, we define the local spaces:

- $PM_{loc}(U) = \{g \in \mathcal{D}'(U) : \varphi g \in PM(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $PF_{loc}(U) = \{g \in \mathcal{D}'(U) : \varphi g \in PF(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $L^1_{loc}(U) \subset PF_{loc}(U)$.
- Every Radon measure on U is a local pseudomeasure.

Let G be analytic on $\Re e\, z > \alpha$ and $U \subset \mathbb{R}$ be open.

We say that G has local pseudofunction boundary behavior on $\alpha + iU$ if it has distributional boundary values there, i.e.

$$\lim_{x \to \alpha^+} G(x + iy) = g(y) \text{ in } \mathcal{D}'(U)$$

and $g \in PF_{loc}(U)$.

Analogously, local pseudomeasure boundary behavior.

Pseudofunctions and pseudomeasures

Pseudofunctions and pseudomeasures are notions that naturally arise in harmonic analysis.

- Pseudomeasures: $PM(\mathbb{R}) = \{g \in \mathcal{S}'(\mathbb{R}) : \widehat{g} \in L^{\infty}(\mathbb{R})\}$
- Pseudofunctions: $PF(\mathbb{R}) = \{g \in PM(\mathbb{R}) : \lim_{x\to\infty} \widehat{g}(x) = 0\}$

Given an open set $U \subseteq \mathbb{R}$, we define the local spaces:

- $PM_{loc}(U) = \{g \in \mathcal{D}'(U) : \varphi g \in PM(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $PF_{loc}(U) = \{g \in \mathcal{D}'(U) : \varphi g \in PF(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $L^1_{loc}(U) \subset PF_{loc}(U)$.
- Every Radon measure on U is a local pseudomeasure.

Let G be analytic on $\Re e\, z > \alpha$ and $U \subset \mathbb{R}$ be open.

We say that G has local pseudofunction boundary behavior on $\alpha + iU$ if it has distributional boundary values there, i.e.

$$\lim_{x\to\alpha^+} G(x+iy) = g(y) \text{ in } \mathcal{D}'(U)$$

and $g \in PF_{loc}(U)$.

Analogously, local pseudomeasure boundary behavior.

Pseudofunctions and pseudomeasures

Pseudofunctions and pseudomeasures are notions that naturally arise in harmonic analysis.

- Pseudomeasures: $PM(\mathbb{R}) = \{g \in \mathcal{S}'(\mathbb{R}) : \widehat{g} \in L^{\infty}(\mathbb{R})\}$
- Pseudofunctions: $PF(\mathbb{R}) = \{g \in PM(\mathbb{R}) : \lim_{x\to\infty} \widehat{g}(x) = 0\}$

Given an open set $U \subseteq \mathbb{R}$, we define the local spaces:

- $PM_{loc}(U) = \{g \in \mathcal{D}'(U) : \varphi g \in PM(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $PF_{loc}(U) = \{g \in \mathcal{D}'(U) : \varphi g \in PF(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $L^1_{loc}(U) \subset PF_{loc}(U)$.
- Every Radon measure on U is a local pseudomeasure.

Let G be analytic on $\Re e\, z > \alpha$ and $U \subset \mathbb{R}$ be open.

We say that G has local pseudofunction boundary behavior on $\alpha + iU$ if it has distributional boundary values there, i.e.

$$\lim_{x\to\alpha^+} G(x+iy) = g(y) \text{ in } \mathcal{D}'(U)$$

and $g \in PF_{loc}(U)$.

Analogously, local pseudomeasure boundary behavior.

Pseudofunctions and pseudomeasures

Pseudofunctions and pseudomeasures are notions that naturally arise in harmonic analysis.

- Pseudomeasures: $PM(\mathbb{R}) = \{g \in \mathcal{S}'(\mathbb{R}) : \widehat{g} \in L^{\infty}(\mathbb{R})\}$
- Pseudofunctions: $PF(\mathbb{R}) = \{g \in PM(\mathbb{R}) : \lim_{x\to\infty} \widehat{g}(x) = 0\}$

Given an open set $U \subseteq \mathbb{R}$, we define the local spaces:

- $PM_{loc}(U) = \{g \in \mathcal{D}'(U) : \varphi g \in PM(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $PF_{loc}(U) = \{g \in \mathcal{D}'(U) : \varphi g \in PF(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $L^1_{loc}(U) \subset PF_{loc}(U)$.
- Every Radon measure on U is a local pseudomeasure.

Let G be analytic on $\Re e\, z > \alpha$ and $U \subset \mathbb{R}$ be open.

We say that G has local pseudofunction boundary behavior on $\alpha + iU$ if it has distributional boundary values there, i.e.

$$\lim_{x\to\alpha^+} G(x+iy) = g(y) \text{ in } \mathcal{D}'(U)$$

and $g \in PF_{loc}(U)$.

Analogously, local pseudomeasure boundary behavior.

Pseudofunctions and pseudomeasures

Pseudofunctions and pseudomeasures are notions that naturally arise in harmonic analysis.

- Pseudomeasures: $PM(\mathbb{R}) = \{g \in \mathcal{S}'(\mathbb{R}) : \ \widehat{g} \in L^{\infty}(\mathbb{R})\}$
- Pseudofunctions: $PF(\mathbb{R}) = \{g \in PM(\mathbb{R}) : \ \lim_{x\to\infty} \widehat{g}(x) = 0\}$

Given an open set $U \subseteq \mathbb{R}$, we define the local spaces:

- $PM_{loc}(U) = \{g \in \mathcal{D}'(U) : \ \varphi g \in PM(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $PF_{loc}(U) = \{g \in \mathcal{D}'(U) : \ \varphi g \in PF(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $L^1_{loc}(U) \subset PF_{loc}(U)$.
- Every Radon measure on U is a local pseudomeasure.

Let G be analytic on $\Re e\, z > \alpha$ and $U \subset \mathbb{R}$ be open.

We say that G has local pseudofunction boundary behavior on $\alpha + iU$ if it has distributional boundary values there, i.e.

$$\lim_{x\to\alpha^+} G(x+iy) = g(y) \text{ in } \mathcal{D}'(U)$$

and $g \in PF_{loc}(U)$.

Analogously, local pseudomeasure boundary behavior.

Extension of the Ingham-Fatou-Riesz theorem

Theorem (Debruyne and Vindas, 2016)

Let $\tau \in L^1_{loc}(\mathbb{R})$ be slowly decreasing, vanish on $(-\infty, 0)$, and have convergent Laplace transform on $\Re e\, z > 0$. Suppose that there is a closed null set $E \subset \mathbb{R}$ such that:

(I) *The analytic function $\mathcal{L}\{\tau; z\} - \sum_{n=1}^{N} \frac{b_n}{z - it_n}$, where $t_n \in \mathbb{R}$, has local pseudofunction boundary behavior $i(\mathbb{R} \setminus E)$,*

(II) *for each $t \in E$ there is $M_t > 0$ such that*

$$\sup_{x>0} \left| \int_0^x \tau(u) e^{-itu} \mathrm{d}u \right| < M_t,$$

(III) $E \cap \{t_1, \dots, t_N\} = \emptyset$.

Then $\tau(x) = \sum_{n=1}^{N} e^{t_n x} + o(1)$.

Theorem (Debruyne and Vindas, 2016)

Let $\tau \in L^1_{loc}(\mathbb{R})$ be slowly decreasing, vanish on $(-\infty, 0)$, and have convergent Laplace transform on $\Re e\, z > 0$. Suppose that there is a closed null set $E \subset \mathbb{R}$ such that:

(I) *The analytic function $\mathcal{L}\{\tau; z\} - \sum_{n=1}^{N} \frac{b_n}{z - it_n}$, where $t_n \in \mathbb{R}$, has local pseudofunction boundary behavior $i(\mathbb{R} \setminus E)$,*

(II) *for each $t \in E$ there is $M_t > 0$ such that*

$$\sup_{x>0} \left| \int_0^x \tau(u) e^{-itu} \mathrm{d}u \right| < M_t,$$

(III) $E \cap \{t_1, \dots, t_N\} = \emptyset$.

Then $\tau(x) = \sum_{n=1}^{N} e^{t_n x} + o(1)$.

Conversely, the relation

$$\tau(x) = \sum_{n=1}^{N} e^{t_n x} + o(1)$$

implies that

$$\mathcal{L}\{\tau; z\} - \sum_{n=1}^{N} \frac{b_n}{z - it_n}$$

has local pseudofunction boundary behavior in the whole imaginary axis $i\mathbb{R}$.

Remark: This shows that there are actually no singular points for the local pseudofunction boundary behavior of $\mathcal{L}\{\tau; z\}$ in the above theorem.

Conversely, the relation

$$\tau(x) = \sum_{n=1}^{N} e^{t_n x} + o(1)$$

implies that

$$\mathcal{L}\{\tau; z\} - \sum_{n=1}^{N} \frac{b_n}{z - it_n}$$

has local pseudofunction boundary behavior in the whole imaginary axis $i\mathbb{R}$.

Remark: This shows that there are actually no singular points for the local pseudofunction boundary behavior of $\mathcal{L}\{\tau; z\}$ in the above theorem.

Second version of the Ingham-Fatou-Riesz theorem

Theorem (Debruyne and Vindas, 2016)

Let $\tau \in L^1_{loc}(\mathbb{R})$ *be slowly decreasing, vanish on* $(-\infty, 0)$*, and have convergent Laplace transform on* $\Re e\, z > 0$*. Let* $\beta_1 \leq \cdots \leq \beta_m \in [0, 1)$ *and* $k_1, \dots, k_m \in \mathbb{Z}_+$*. The analytic function*

$$\mathcal{L}\{\tau; z\} - \frac{a}{z^2} - \sum_{n=1}^{N} \frac{b_n}{z - it_n} - \sum_{n=1}^{m} \frac{c_n + d_n \log^{k_n}(1/z)}{z^{\beta_n + 1}} \qquad (t_n \in \mathbb{R})$$

has local pseudofunction boundary behavior on $\Re e\, z = 0$ *if and only if*

$$\tau(x) = ax + \sum_{n=1}^{N} b_n e^{it_n x} + \sum_{n=1}^{m} \frac{c_n x^{\beta_n}}{\Gamma(\beta_n + 1)} + \sum_{n=1}^{m} d_n x^{\beta_n} \sum_{j=0}^{k_n} \binom{k_n}{j} D_j(\beta_n + 1) \log^{k_n - j} x + o(1),$$

where $D_j(\omega) = \frac{d^j}{dy^j}\left(\frac{1}{\Gamma(y)}\right)\bigg|_{y=\omega}$.

Second version of the Ingham-Fatou-Riesz theorem

Theorem (Debruyne and Vindas, 2016)

Let $\tau \in L^1_{loc}(\mathbb{R})$ *be slowly decreasing, vanish on* $(-\infty, 0)$*, and have convergent Laplace transform on* $\Re e\, z > 0$*. Let* $\beta_1 \leq \cdots \leq \beta_m \in [0, 1)$ *and* $k_1, \ldots, k_m \in \mathbb{Z}_+$*. The analytic function*

$$\mathcal{L}\{\tau; z\} - \frac{a}{z^2} - \sum_{n=1}^{N} \frac{b_n}{z - it_n} - \sum_{n=1}^{m} \frac{c_n + d_n \log^{k_n}(1/z)}{z^{\beta_n + 1}} \qquad (t_n \in \mathbb{R})$$

has local pseudofunction boundary behavior on $\Re e\, z = 0$ *if and only if*

$$\tau(x) = ax + \sum_{n=1}^{N} b_n e^{it_n x} + \sum_{n=1}^{m} \frac{c_n x^{\beta_n}}{\Gamma(\beta_n + 1)} + \sum_{n=1}^{m} d_n x^{\beta_n} \sum_{j=0}^{k_n} \binom{k_n}{j} D_j(\beta_n + 1) \log^{k_n - j} x + o(1),$$

where $D_j(\omega) = \frac{d^j}{dy^j}\left(\frac{1}{\Gamma(y)}\right)\Big|_{y=\omega}$.

Extension of the Korevaar-Wiener-Ikehara theorem

Theorem (Debruyne and Vindas, 2016)

Let S be a non-decreasing function and supported in $[0,\infty)$ such that $\mathcal{L}\{\mathrm{d}S; z\} = \int_{0^-}^{\infty} e^{-zt}\mathrm{d}S(t)$ converges for $\Re e\, z > \alpha > 0$.
Suppose that there are a closed null set E, constants $r_0, r_1, \dots, r_N \in \mathbb{R}$, $\theta_1, \dots, \theta_N \in \mathbb{R}$, and $t_1, \dots, t_N > 0$ such that:

(I) $\mathcal{L}\{\mathrm{d}S; z\} - \frac{r_0}{z-\alpha} - \sum_{n=1}^{N} r_n \left(\frac{e^{i\theta_n}}{z-\alpha-it_n} + \frac{e^{-i\theta_n}}{z-\alpha+it_n} \right)$
admits local pseudofunction boundary behavior on $\alpha + i(\mathbb{R}\setminus E)$,

(II) $E \cap \{0, t_1, \dots, t_N\} = \emptyset$, *and*

(III) *for every $t \in E$, $\int_0^x e^{-\alpha u - itu}\mathrm{d}S(u) = O_t(1)$.*

Then

$$S(x) = e^{\alpha x}\left(\frac{r_0}{\alpha} + 2\sum_{n=1}^{N} \frac{r_n \cos(t_n x + \theta_n - \arctan(t_n/\alpha))}{\sqrt{\alpha^2 + t_n^2}} + o(1) \right).$$

Extension of the Korevaar-Wiener-Ikehara theorem

Theorem (Debruyne and Vindas, 2016)

Let S be a non-decreasing function and supported in $[0,\infty)$ such that $\mathcal{L}\{\mathrm{d}S;z\} = \int_{0^-}^{\infty} e^{-zt}\mathrm{d}S(t)$ converges for $\Re e\, z > \alpha > 0$.
Suppose that there are a closed null set E, constants $r_0, r_1, \dots, r_N \in \mathbb{R}$, $\theta_1, \dots, \theta_N \in \mathbb{R}$, and $t_1, \dots, t_N > 0$ such that:

(I) $$\mathcal{L}\{\mathrm{d}S;z\} - \frac{r_0}{z-\alpha} - \sum_{n=1}^{N} r_n \left(\frac{e^{i\theta_n}}{z-\alpha-it_n} + \frac{e^{-i\theta_n}}{z-\alpha+it_n} \right)$$
admits local pseudofunction boundary behavior on $\alpha + i(\mathbb{R}\setminus E)$,

(II) $E \cap \{0, t_1, \dots, t_N\} = \emptyset$, *and*

(III) *for every $t \in E$,* $\int_0^x e^{-\alpha u - itu}\mathrm{d}S(u) = O_t(1)$.

Then

$$S(x) = e^{\alpha x}\left(\frac{r_0}{\alpha} + 2\sum_{n=1}^{N} \frac{r_n \cos(t_n x + \theta_n - \arctan(t_n/\alpha))}{\sqrt{\alpha^2 + t_n^2}} + o(1) \right).$$

Extension of the Korevaar-Wiener-Ikehara theorem

Theorem (Debruyne and Vindas, 2016)

Let S be a non-decreasing function and supported in $[0,\infty)$ such that $\mathcal{L}\{\mathrm{d}S;z\}=\int_{0^-}^{\infty}e^{-zt}\mathrm{d}S(t)$ converges for $\Re e\, z>\alpha>0$.
Suppose that there are a closed null set E, constants $r_0,r_1,\dots,r_N\in\mathbb{R}$, $\theta_1,\dots,\theta_N\in\mathbb{R}$, and $t_1,\dots,t_N>0$ such that:

(I) $$\mathcal{L}\{\mathrm{d}S;z\}-\frac{r_0}{z-\alpha}-\sum_{n=1}^{N}r_n\left(\frac{e^{i\theta_n}}{z-\alpha-it_n}+\frac{e^{-i\theta_n}}{z-\alpha+it_n}\right)$$
admits local pseudofunction boundary behavior on $\alpha+i(\mathbb{R}\setminus E)$,

(II) $E\cap\{0,t_1,\dots,t_N\}=\emptyset$, *and*

(III) *for every $t\in E$, $\int_0^x e^{-\alpha u-itu}\mathrm{d}S(u)=O_t(1)$.*

Then

$$S(x)=e^{\alpha x}\left(\frac{r_0}{\alpha}+2\sum_{n=1}^{N}\frac{r_n\cos(t_nx+\theta_n-\arctan(t_n/\alpha))}{\sqrt{\alpha^2+t_n^2}}+o(1)\right).$$

Conversely, if S has asymptotic behavior

$$S(x) = e^{\alpha x}\left(\frac{r_0}{\alpha} + 2\sum_{n=1}^{N}\frac{r_n\cos(t_n x + \theta_n - \arctan(t_n/\alpha))}{\sqrt{\alpha^2 + t_n^2}} + o(1)\right).$$

then

$$\mathcal{L}\{\mathrm{d}S; z\} - \frac{r_0}{z-\alpha} - \sum_{n=1}^{N} r_n\left(\frac{e^{i\theta_n}}{z-\alpha-it_n} + \frac{e^{-i\theta_n}}{z-\alpha+it_n}\right)$$

has local pseudofunction boundary behavior on the whole line $\Re e\, z = \alpha$.

Extension of the Katznelson-Tzafriri theorem

Theorem (Debruyne and Vindas, 2016)

Let $F(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$. Suppose that there is a closed null subset $E \subset \partial\mathbb{D}$ such that F has local pseudofunction boundary behavior on $\partial\mathbb{D} \setminus E$, whereas for each $e^{i\theta} \in E$

$$\sum_{n=0}^{N} c_n e^{in\theta} = O_\theta(1)$$

Then, $c_n = o(1)$. Moreover, the series $\sum_{n=0}^{\infty} c_n e^{in\theta_0}$ converges at every point where there is a constant $F(e^{i\theta_0})$ such that

$$\frac{F(z) - F(e^{i\theta_0})}{z - e^{i\theta_0}}$$

has pseudofunction boundary behavior at $z = e^{i\theta_0} \in \partial\mathbb{D}$, and

$$\sum_{n=0}^{\infty} c_n e^{in\theta_0} = F(e^{i\theta_0}).$$

Extension of the Katznelson-Tzafriri theorem

Theorem (Debruyne and Vindas, 2016)

Let $F(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$. Suppose that there is a closed null subset $E \subset \partial\mathbb{D}$ such that F has local pseudofunction boundary behavior on $\partial\mathbb{D} \setminus E$, whereas for each $e^{i\theta} \in E$

$$\sum_{n=0}^{N} c_n e^{in\theta} = O_\theta(1)$$

Then, $c_n = o(1)$. Moreover, the series $\sum_{n=0}^{\infty} c_n e^{in\theta_0}$ converges at every point where there is a constant $F(e^{i\theta_0})$ such that

$$\frac{F(z) - F(e^{i\theta_0})}{z - e^{i\theta_0}}$$

has pseudofunction boundary behavior at $z = e^{i\theta_0} \in \partial\mathbb{D}$, and

$$\sum_{n=0}^{\infty} c_n e^{in\theta_0} = F(e^{i\theta_0}).$$

An important particular case

Showing all of the above four theorems may be reduced to:

Theorem

Let $\tau \in L^1_{loc}(\mathbb{R})$ be slowly decreasing, vanish on $(-\infty, 0)$, and have convergent Laplace transform on $\Re e\, z > 0$. Suppose there is a closed null set $E \subset \mathbb{R}$ such that:

(I) *$\mathcal{L}\{\tau; z\}$ has local pseudofunction boundary behavior on $i(\mathbb{R} \setminus E)$,*

(II) *for each $t \in E$ there is $M_t > 0$ such that*

$$\sup_{x>0} \left| \int_0^x \tau(u) e^{-itu} \mathrm{d}u \right| < M_t,$$

(III) *$0 \notin E$.*

Then

$$\tau(x) = o(1).$$

An important particular case

Showing all of the above four theorems may be reduced to:

Theorem

Let $\tau \in L^1_{loc}(\mathbb{R})$ be slowly decreasing, vanish on $(-\infty, 0)$, and have convergent Laplace transform on $\Re e\, z > 0$. Suppose there is a closed null set $E \subset \mathbb{R}$ such that:

(I) *$\mathcal{L}\{\tau; z\}$ has local pseudofunction boundary behavior on $i(\mathbb{R} \setminus E)$,*

(II) *for each $t \in E$ there is $M_t > 0$ such that*

$$\sup_{x>0} \left| \int_0^x \tau(u) e^{-itu} \mathrm{d}u \right| < M_t,$$

(III) *$0 \notin E$.*

Then

$$\tau(x) = o(1).$$

Some tools

Our approach is based in the following tools:

1. Boundedness theorems (crucial)
2. A characterization of local pseudofunctions (also crucial)
3. Distributional methods (standard)

The next notion plays a key role for boundedness theorems:

A function τ is boundedly decreasing if there is a $\delta > 0$ such that

$$\liminf_{x\to\infty} \inf_{h\in[0,\delta]} (\tau(x+h) - \tau(x)) > -\infty,$$

that is, if there are constants $\delta, X, M > 0$ such that

$$\tau(x+h) - \tau(x) \geq -M, \quad \text{for } 0 \leq h \leq \delta \text{ and } x \geq X.$$

Some tools

Our approach is based in the following tools:

1. Boundedness theorems (crucial)
2. A characterization of local pseudofunctions (also crucial)
3. Distributional methods (standard)

The next notion plays a key role for boundedness theorems:

A function τ is boundedly decreasing if there is a $\delta > 0$ such that

$$\liminf_{x\to\infty} \inf_{h\in[0,\delta]} (\tau(x+h) - \tau(x)) > -\infty,$$

that is, if there are constants $\delta, X, M > 0$ such that

$$\tau(x+h) - \tau(x) \geq -M, \quad \text{for } 0 \leq h \leq \delta \text{ and } x \geq X.$$

Boundedness theorem

Our main boundedness result allows us to conclude boundedness of a boundedly decreasing function from the boundary behavior of its Laplace transform at $z = 0$.

Theorem (Debruyne and Vindas, 2016)

Let $\tau \in L^1_{loc}(\mathbb{R})$ vanish on $(-\infty, 0)$ and have convergent Laplace transform $\mathcal{L}\{\tau; z\} = \int_0^\infty \tau(t)e^{-zt}\mathrm{d}t$ for $\Re e\, z > 0$. Suppose the following Tauberian conditions is satisfied:

$$\tau \text{ is boundedly decreasing.}$$

If $\mathcal{L}\{\tau; z\}$ has local pseudomeasure boundary behavior at $z = 0$ (i.e. in some imaginary segment $i(-\lambda, \lambda)$), then

$$\tau(x) = O(1), \quad x \to \infty.$$

The proof of this result is involved and technical. It generalizes an earlier boundedness theorem stated by Korevaar.

Boundedness theorem

Our main boundedness result allows us to conclude boundedness of a boundedly decreasing function from the boundary behavior of its Laplace transform at $z = 0$.

Theorem (Debruyne and Vindas, 2016)

Let $\tau \in L^1_{loc}(\mathbb{R})$ vanish on $(-\infty, 0)$ and have convergent Laplace transform $\mathcal{L}\{\tau; z\} = \int_0^\infty \tau(t)e^{-zt}\mathrm{d}t$ for $\Re e\, z > 0$. Suppose the following Tauberian conditions is satisfied:

$$\tau \text{ is boundedly decreasing.}$$

If $\mathcal{L}\{\tau; z\}$ has local pseudomeasure boundary behavior at $z = 0$ (i.e. in some imaginary segment $i(-\lambda, \lambda)$), then

$$\tau(x) = O(1), \quad x \to \infty.$$

The proof of this result is involved and technical. It generalizes an earlier boundedness theorem stated by Korevaar.

Characterization of local pseudofunctions and the pseudofunction singular support of distributions

We introduce:

Given $f \in \mathcal{D}'(U)$, its singular pseudofunction support in U, denoted as $\operatorname{sing\,supp}_{PF} f$, is the complement in U of the largest open subset of U where f is a local pseudofunction.

Theorem (Debruyne and Vindas, 2016)

Let $f \in \mathcal{D}'(U)$. Suppose there is a closed null set $E \subset U$ such that

(I) $\operatorname{sing\,supp}_{PF} f \subseteq E$, *and*

(II) *for each $t_0 \in E$ there is a neighborhood V_{t_0} of t_0 and a local pseudomeasure $f_{t_0} \in PM_{loc}(V_{t_0})$ such that*

$$f = (t - t_0) f_{t_0} \quad \text{on } V_{t_0}.$$

Then, $\operatorname{sing\,supp}_{PF} f = \emptyset$, that is, $f \in PF_{loc}(U)$.

Characterization of local pseudofunctions and the pseudofunction singular support of distributions

We introduce:

Given $f \in \mathcal{D}'(U)$, its singular pseudofunction support in U, denoted as $\operatorname{sing\,supp}_{PF} f$, is the complement in U of the largest open subset of U where f is a local pseudofunction.

Theorem (Debruyne and Vindas, 2016)

Let $f \in \mathcal{D}'(U)$. Suppose there is a closed null set $E \subset U$ such that

(I) $\operatorname{sing\,supp}_{PF} f \subseteq E$, *and*

(II) *for each $t_0 \in E$ there is a neighborhood V_{t_0} of t_0 and a local pseudomeasure $f_{t_0} \in PM_{loc}(V_{t_0})$ such that*

$$f = (t - t_0) f_{t_0} \quad \text{on } V_{t_0} .$$

Then, $\operatorname{sing\,supp}_{PF} f = \emptyset$, *that is,* $f \in PF_{loc}(U)$.

Given $g \in \mathcal{S}'(\mathbb{R})$, we define its pseudofunction spectrum as the closed set $\operatorname{sp}_{PF}(g) = \operatorname{sing\,supp}_{PF} \widehat{g}$.

The space of bounded distributions $\mathcal{B}'(\mathbb{R})$ is the dual of

$$\mathcal{D}_{L^1}(\mathbb{R}) = \{\varphi \in C^\infty(\mathbb{R}) |\, \varphi^{(n)} \in L^1(\mathbb{R}),\ \forall n \in \mathbb{N}\}.$$

$\dot{\mathcal{B}}'(\mathbb{R})$, the space of distributions 'vanishing' at $\pm\infty$, is the completion of $\mathcal{D}(\mathbb{R})$ in (the strong topology of) $\mathcal{B}'(\mathbb{R})$.

Lemma

Let $\tau \in \mathcal{B}'(\mathbb{R})$. Then, $\tau \in \dot{\mathcal{B}}'(\mathbb{R})$ if and only if $\operatorname{sp}_{PF}(\tau) = \emptyset$.

Lemma

Let $\tau \in \dot{\mathcal{B}}'(\mathbb{R}) \cap L^1_{loc}(\mathbb{R})$ be slowly decreasing. Then

$$\lim_{x\to\infty} \tau(x) = 0.$$

Pseudofunction spectrum

Given $g \in \mathcal{S}'(\mathbb{R})$, we define its pseudofunction spectrum as the closed set $\operatorname{sp}_{PF}(g) = \operatorname{sing\,supp}_{PF} \widehat{g}$.

The space of bounded distributions $\mathcal{B}'(\mathbb{R})$ is the dual of

$$\mathcal{D}_{L^1}(\mathbb{R}) = \{\varphi \in C^{\infty}(\mathbb{R}) |\, \varphi^{(n)} \in L^1(\mathbb{R}),\ \forall n \in \mathbb{N}\}.$$

$\dot{\mathcal{B}}'(\mathbb{R})$, the space of distributions 'vanishing' at $\pm\infty$, is the completion of $\mathcal{D}(\mathbb{R})$ in (the strong topology of) $\mathcal{B}'(\mathbb{R})$.

Lemma

Let $\tau \in \mathcal{B}'(\mathbb{R})$. Then, $\tau \in \dot{\mathcal{B}}'(\mathbb{R})$ if and only if $\operatorname{sp}_{PF}(\tau) = \emptyset$.

Lemma

Let $\tau \in \dot{\mathcal{B}}'(\mathbb{R}) \cap L^1_{loc}(\mathbb{R})$ be slowly decreasing. Then

$$\lim_{x \to \infty} \tau(x) = 0.$$

Back to the Tauberian theorem

Theorem

$\tau \in L^1_{loc}(\mathbb{R})$ slowly decreasing with convergent $\mathcal{L}\{\tau; z\}$ on $\Re e\, z > 0$. A closed null set $E \subset \mathbb{R}$ such that:

(I) *$\mathcal{L}\{\tau; z\}$ has local pseudofunction boundary behavior on $i(\mathbb{R} \setminus E)$,*

(II) *for each $t \in E$ there is $M_t > 0$ such that*

$$\sup_{x>0} \left| \int_0^x \tau(u) e^{-itu} \mathrm{d}u \right| < M_t,$$

(III) *$0 \notin E$.*

Then $\tau(x) = o(1)$.

- (III) $\Rightarrow \tau$ is bounded near ∞ (boundedness theorem) $\Rightarrow$ $\tau \in \mathcal{B}'(\mathbb{R})$.
- (I) and (II) $\Rightarrow \hat{\tau} \in PF_{loc}(\mathbb{R})$ (characterization of pseudofunctions).
- $\operatorname{sp}_{PF}(\tau) = \operatorname{singsupp}_{PF} \hat{\tau} = \emptyset$ and $\tau \in \mathcal{B}'(\mathbb{R}) \Rightarrow \tau \in \dot{\mathcal{B}}'(\mathbb{R})$.
- $\tau \in \dot{\mathcal{B}}'(\mathbb{R})$ and slowly decreasing $\Rightarrow \lim_{x\to\infty} \tau(x) = 0$.

Back to the Tauberian theorem

Theorem

$\tau \in L^1_{loc}(\mathbb{R})$ slowly decreasing with convergent $\mathcal{L}\{\tau; z\}$ on $\Re e\, z > 0$. A closed null set $E \subset \mathbb{R}$ such that:

(I) *$\mathcal{L}\{\tau; z\}$ has local pseudofunction boundary behavior on $i(\mathbb{R} \setminus E)$,*

(II) *for each $t \in E$ there is $M_t > 0$ such that*

$$\sup_{x>0} \left| \int_0^x \tau(u) e^{-itu} \mathrm{d}u \right| < M_t,$$

(III) *$0 \notin E$.*

Then $\tau(x) = o(1)$.

- (III) $\Rightarrow$ τ is bounded near ∞ (boundedness theorem) $\Rightarrow$ $\tau \in \mathcal{B}'(\mathbb{R})$.
- (I) and (II) $\Rightarrow$ $\hat{\tau} \in PF_{loc}(\mathbb{R})$ (characterization of pseudofunctions).
- $\operatorname{sp}_{PF}(\tau) = \operatorname{singsupp}_{PF} \hat{\tau} = \emptyset$ and $\tau \in \mathcal{B}'(\mathbb{R}) \Rightarrow \tau \in \dot{\mathcal{B}}'(\mathbb{R})$.
- $\tau \in \dot{\mathcal{B}}'(\mathbb{R})$ and slowly decreasing $\Rightarrow \lim_{x\to\infty} \tau(x) = 0$.

Back to the Tauberian theorem

Theorem

$\tau \in L^1_{loc}(\mathbb{R})$ slowly decreasing with convergent $\mathcal{L}\{\tau; z\}$ on $\Re e\, z > 0$. A closed null set $E \subset \mathbb{R}$ such that:

(I) *$\mathcal{L}\{\tau; z\}$ has local pseudofunction boundary behavior on $i(\mathbb{R} \setminus E)$,*

(II) *for each $t \in E$ there is $M_t > 0$ such that*

$$\sup_{x>0} \left| \int_0^x \tau(u) e^{-itu} \mathrm{d}u \right| < M_t,$$

(III) *$0 \notin E$.*

Then $\tau(x) = o(1)$.

- (III) $\Rightarrow$ τ is bounded near ∞ (boundedness theorem) $\Rightarrow$ $\tau \in \mathcal{B}'(\mathbb{R})$.
- (I) and (II) $\Rightarrow$ $\widehat{\tau} \in PF_{loc}(\mathbb{R})$ (characterization of pseudofunctions).
- $\mathrm{sp}_{PF}(\tau) = \mathrm{singsupp}_{PF}\, \widehat{\tau} = \emptyset$ and $\tau \in \mathcal{B}'(\mathbb{R}) \Rightarrow \tau \in \dot{\mathcal{B}}'(\mathbb{R})$.
- $\tau \in \dot{\mathcal{B}}'(\mathbb{R})$ and slowly decreasing $\Rightarrow \lim_{x\to\infty} \tau(x) = 0$.

Back to the Tauberian theorem

Theorem

$\tau \in L^1_{loc}(\mathbb{R})$ slowly decreasing with convergent $\mathcal{L}\{\tau; z\}$ on $\Re e\, z > 0$. A closed null set $E \subset \mathbb{R}$ such that:

(I) *$\mathcal{L}\{\tau; z\}$ has local pseudofunction boundary behavior on $i(\mathbb{R} \setminus E)$,*

(II) *for each $t \in E$ there is $M_t > 0$ such that*

$$\sup_{x>0} \left| \int_0^x \tau(u) e^{-itu} \mathrm{d}u \right| < M_t,$$

(III) *$0 \notin E$.*

Then $\tau(x) = o(1)$.

- (III) $\Rightarrow$ τ is bounded near ∞ (boundedness theorem) $\Rightarrow$ $\tau \in \mathcal{B}'(\mathbb{R})$.
- (I) and (II) $\Rightarrow \widehat{\tau} \in PF_{loc}(\mathbb{R})$ (characterization of pseudofunctions).
- $\mathrm{sp}_{PF}(\tau) = \mathrm{singsupp}_{PF}\, \widehat{\tau} = \emptyset$ and $\tau \in \mathcal{B}'(\mathbb{R}) \Rightarrow \tau \in \dot{\mathcal{B}}'(\mathbb{R})$.
- $\tau \in \dot{\mathcal{B}}'(\mathbb{R})$ and slowly decreasing $\Rightarrow \lim_{x\to\infty} \tau(x) = 0$.

Some references

The last part of this talk is based on our recent work:

- G. Debruyne, J. Vindas, Complex Tauberian theorems for Laplace transforms with local pseudofunction boundary behavior, J. Anal. Math., to appear (preprint: arXiv:1604.05069)

For applications of these recent results in analytic number theory, see:

- G. Debruyne, J. Vindas, On PNT equivalences for Beurling numbers, Monatsh. Math., to appear (preprint arXiv:1606.03579)

Important book references on Tauberians

- W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander, Vector-valued Laplace transforms and Cauchy problems, Birkhäuser/Springer Basel, 2011
- J. Korevaar, Tauberian theory. A century of developments, Springer-Verlag, 2004

Some references

The last part of this talk is based on our recent work:

- G. Debruyne, J. Vindas, Complex Tauberian theorems for Laplace transforms with local pseudofunction boundary behavior, J. Anal. Math., to appear (preprint: arXiv:1604.05069)

For applications of these recent results in analytic number theory, see:

- G. Debruyne, J. Vindas, On PNT equivalences for Beurling numbers, Monatsh. Math., to appear (preprint arXiv:1606.03579)

Important book references on Tauberians

- W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander, Vector-valued Laplace transforms and Cauchy problems, Birkhäuser/Springer Basel, 2011
- J. Korevaar, Tauberian theory. A century of developments, Springer-Verlag, 2004