The nuclearity of Gelfand-Shilov spaces and kernel theorems

Jasson Vindas
jasson.vindas@UGent.be

Ghent University

13th International ISAAC Congress
Session Generalized Functions and Applications
Ghent, August 2, 2021

IWOTA 2021
Session Pseudo-differential Operators
Orange, August 11, 2021

Nuclear spaces play a major role in functional analysis.

Key feature: the validity of abstract Schwartz kernel theorems

Establishing nuclearity for a function space: central question from the point of view of applications and understanding its locally convex structure.

In this talk we discuss:

- Nuclearity for global spaces of ultradifferentiable functions with controlled decay at infinity.
- These spaces are of Gelfand-Shilov type.
- Our results are counterparts of characterizations of
    - nuclear Köthe sequence spaces;
    - nuclear Gelfand-Shilov spaces of smooth functions.
- We obtain new kernel theorems in this context.

The talk is based on collaborative works with Andreas Debrouwere and Lenny Neyt.

Nuclear spaces play a major role in functional analysis.

Key feature: the validity of abstract Schwartz kernel theorems

Establishing nuclearity for a function space: central question from the point of view of applications and understanding its locally convex structure.

In this talk we discuss:

- Nuclearity for global spaces of ultradifferentiable functions with controlled decay at infinity.
- These spaces are of Gelfand-Shilov type.
- Our results are counterparts of characterizations of
  - nuclear Köthe sequence spaces;
  - nuclear Gelfand-Shilov spaces of smooth functions.
- We obtain new kernel theorems in this context.

The talk is based on collaborative works with Andreas Debrouwere and Lenny Neyt.

Nuclear spaces play a major role in functional analysis.

Key feature: the validity of abstract Schwartz kernel theorems

Establishing nuclearity for a function space: central question from the point of view of applications and understanding its locally convex structure.

In this talk we discuss:

- Nuclearity for global spaces of ultradifferentiable functions with controlled decay at infinity.
- These spaces are of Gelfand-Shilov type.
- Our results are counterparts of characterizations of
  - nuclear Köthe sequence spaces;
  - nuclear Gelfand-Shilov spaces of smooth functions.
- We obtain new kernel theorems in this context.

The talk is based on collaborative works with Andreas Debrouwere and Lenny Neyt.

Nuclear spaces play a major role in functional analysis.

Key feature: the validity of abstract Schwartz kernel theorems

Establishing nuclearity for a function space: central question from the point of view of applications and understanding its locally convex structure.

In this talk we discuss:

- Nuclearity for global spaces of ultradifferentiable functions with controlled decay at infinity.
- These spaces are of Gelfand-Shilov type.
- Our results are counterparts of characterizations of
  - nuclear Köthe sequence spaces;
  - nuclear Gelfand-Shilov spaces of smooth functions.
- We obtain new kernel theorems in this context.

The talk is based on collaborative works with Andreas Debrouwere and Lenny Neyt.

Nuclear spaces play a major role in functional analysis.

Key feature: the validity of abstract Schwartz kernel theorems

Establishing nuclearity for a function space: central question from the point of view of applications and understanding its locally convex structure.

In this talk we discuss:

- Nuclearity for global spaces of ultradifferentiable functions with controlled decay at infinity.
- These spaces are of Gelfand-Shilov type.
- Our results are counterparts of characterizations of
  - nuclear Köthe sequence spaces;
  - nuclear Gelfand-Shilov spaces of smooth functions.
- We obtain new kernel theorems in this context.

The talk is based on collaborative works with Andreas Debrouwere and Lenny Neyt.

Nuclear spaces play a major role in functional analysis.

Key feature: the validity of abstract Schwartz kernel theorems

Establishing nuclearity for a function space: central question from the point of view of applications and understanding its locally convex structure.

In this talk we discuss:

- Nuclearity for global spaces of ultradifferentiable functions with controlled decay at infinity.
- These spaces are of Gelfand-Shilov type.
- Our results are counterparts of characterizations of
  - nuclear Köthe sequence spaces;
  - nuclear Gelfand-Shilov spaces of smooth functions.
- We obtain new kernel theorems in this context.

The talk is based on collaborative works with Andreas Debrouwere and Lenny Neyt.

Nuclear spaces play a major role in functional analysis.

Key feature: the validity of abstract Schwartz kernel theorems

Establishing nuclearity for a function space: central question from the point of view of applications and understanding its locally convex structure.

In this talk we discuss:

- Nuclearity for global spaces of ultradifferentiable functions with controlled decay at infinity.
- These spaces are of Gelfand-Shilov type.
- Our results are counterparts of characterizations of
  - nuclear Köthe sequence spaces;
  - nuclear Gelfand-Shilov spaces of smooth functions.
- We obtain new kernel theorems in this context.

The talk is based on collaborative works with Andreas Debrouwere and Lenny Neyt.

Nuclear spaces play a major role in functional analysis.

Key feature: the validity of abstract Schwartz kernel theorems

Establishing nuclearity for a function space: central question from the point of view of applications and understanding its locally convex structure.

In this talk we discuss:

- Nuclearity for global spaces of ultradifferentiable functions with controlled decay at infinity.
- These spaces are of Gelfand-Shilov type.
- Our results are counterparts of characterizations of
  - nuclear Köthe sequence spaces;
  - nuclear Gelfand-Shilov spaces of smooth functions.
- We obtain new kernel theorems in this context.

The talk is based on collaborative works with Andreas Debrouwere and Lenny Neyt.

Schwartz kernel theorem

Schwartz' kernel theorem: $\mathcal{S}'(\mathbb{R}^{d_1+d_2}) \cong \mathcal{L}(\mathcal{S}(\mathbb{R}^{d_1}), \mathcal{S}'(\mathbb{R}^{d_2}))$

Natural isomorphism: each continuous $L : \mathcal{S}(\mathbb{R}^{d_1}) \to \mathcal{S}'(\mathbb{R}^{d_2})$ is determined by

$$\langle L(\varphi_1), \varphi_2 \rangle = \langle f(x,y), \varphi_1(x)\varphi_2(y) \rangle ,$$

for some distribution kernel $f \in \mathcal{S}'(\mathbb{R}^{d_1+d_2})$

Grothendieck discovered nuclearity is the underlying property of a lcs for the validity of abstract Schwartz kernel theorems.

"With a few exceptions the locally convex spaces encountered in analysis can be divided into two classes. First there are the normed spaces, [....]. The second class consists of the so-called nuclear locally convex spaces ..."

A. Pietsch

Schwartz kernel theorem

Schwartz' kernel theorem: $\mathcal{S}'(\mathbb{R}^{d_1+d_2}) \cong \mathcal{L}(\mathcal{S}(\mathbb{R}^{d_1}), \mathcal{S}'(\mathbb{R}^{d_2}))$

Natural isomorphism: each continuous $L : \mathcal{S}(\mathbb{R}^{d_1}) \to \mathcal{S}'(\mathbb{R}^{d_2})$ is determined by

$$\langle L(\varphi_1), \varphi_2 \rangle = \langle f(x,y), \varphi_1(x)\varphi_2(y) \rangle ,$$

for some distribution kernel $f \in \mathcal{S}'(\mathbb{R}^{d_1+d_2})$

Grothendieck discovered nuclearity is the underlying property of a lcs for the validity of abstract Schwartz kernel theorems.

"With a few exceptions the locally convex spaces encountered in analysis can be divided into two classes. First there are the normed spaces, [....]. The second class consists of the so-called nuclear locally convex spaces ..."

A. Pietsch

Schwartz kernel theorem

Schwartz' kernel theorem: $\mathcal{S}'(\mathbb{R}^{d_1+d_2}) \cong \mathcal{L}(\mathcal{S}(\mathbb{R}^{d_1}), \mathcal{S}'(\mathbb{R}^{d_2}))$

Natural isomorphism: each continuous $L : \mathcal{S}(\mathbb{R}^{d_1}) \to \mathcal{S}'(\mathbb{R}^{d_2})$ is determined by

$$\langle L(\varphi_1), \varphi_2 \rangle = \langle f(x,y), \varphi_1(x)\varphi_2(y) \rangle \,,$$

for some distribution kernel $f \in \mathcal{S}'(\mathbb{R}^{d_1+d_2})$

Grothendieck discovered nuclearity is the underlying property of a lcs for the validity of abstract Schwartz kernel theorems.

"With a few exceptions the locally convex spaces encountered in analysis can be divided into two classes. First there are the normed spaces, [....]. The second class consists of the so-called nuclear locally convex spaces ..."

A. Pietsch

Schwartz kernel theorem

Schwartz' kernel theorem: $\mathcal{S}'(\mathbb{R}^{d_1+d_2}) \cong \mathcal{L}(\mathcal{S}(\mathbb{R}^{d_1}), \mathcal{S}'(\mathbb{R}^{d_2}))$

Natural isomorphism: each continuous $L : \mathcal{S}(\mathbb{R}^{d_1}) \to \mathcal{S}'(\mathbb{R}^{d_2})$ is determined by

$$\langle L(\varphi_1), \varphi_2 \rangle = \langle f(x,y), \varphi_1(x)\varphi_2(y) \rangle \,,$$

for some distribution kernel $f \in \mathcal{S}'(\mathbb{R}^{d_1+d_2})$

Grothendieck discovered nuclearity is the underlying property of a lcs for the validity of abstract Schwartz kernel theorems.

"With a few exceptions the locally convex spaces encountered in analysis can be divided into two classes. First there are the normed spaces, [....]. The second class consists of the so-called nuclear locally convex spaces ..."

A. Pietsch

Nuclear spaces

Nuclear maps

Let E and F be Banach spaces. A nuclear map $L : E \to F$ is a trace-class map, that is, one that is representable as

$$L = \sum_{j=1}^{\infty} \lambda_j (x'_j \otimes y_j) \qquad \text{with } (\lambda_j) \in \ell^1,\ y_j \in F, \text{ and } x_j \in E'.$$

Nuclear space

A lcHs E is nuclear if for every continuous seminorm p there is another one $q \geq p$ such that the natural map $\widehat{E}_q \to \widehat{E}_p$ is nuclear.

Grothendieck's criterion

Let E be either a Fréchet space or a (DF)-space. Then, E is nuclear if and only if every weakly summable sequence in E is absolutely summable.

Nuclear spaces

Nuclear maps

Let E and F be Banach spaces. A nuclear map $L : E \to F$ is a trace-class map, that is, one that is representable as

$$L = \sum_{j=1}^{\infty} \lambda_j (x_j' \otimes y_j) \qquad \text{with } (\lambda_j) \in \ell^1,\ y_j \in F, \text{ and } x_j \in E'.$$

Nuclear space

A lcHs E is nuclear if for every continuous seminorm p there is another one $q \geq p$ such that the natural map $\widehat{E}_q \to \widehat{E}_p$ is nuclear.

Grothendieck's criterion

Let E be either a Fréchet space or a (DF)-space. Then, E is nuclear if and only if every weakly summable sequence in E is absolutely summable.

Nuclear spaces

Nuclear maps

Let E and F be Banach spaces. A nuclear map $L : E \to F$ is a trace-class map, that is, one that is representable as

$$L = \sum_{j=1}^{\infty} \lambda_j (x_j' \otimes y_j) \qquad \text{with } (\lambda_j) \in \ell^1,\ y_j \in F, \text{ and } x_j \in E'.$$

Nuclear space

A lcHs E is nuclear if for every continuous seminorm p there is another one $q \geq p$ such that the natural map $\widehat{E}_q \to \widehat{E}_p$ is nuclear.

Grothendieck's criterion

Let E be either a Fréchet space or a (DF)-space. Then, E is nuclear if and only if every weakly summable sequence in E is absolutely summable.

- Consider $\mathscr{W} = (w_n)_{n\in\mathbb{N}}$ with $w_n \in C(\mathbb{R}^d)$ and $1 \leq w_1 \leq w_2 \leq \cdots$.
- $\mathcal{K}(\mathscr{W}) = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \max_{|\alpha|\leq n} \|\varphi^{(\alpha)} w_n\|_{L^\infty} < \infty \quad \forall n \in \mathbb{N}\}$.

Theorem

Assume the mild regularity hypothesis: $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N}\, \exists C > 0$

$$\sup_{|y|\leq 1} w_n(x+y) \leq C w_m(x), \quad \forall x \in \mathbb{R}^d.$$

Then, $\mathcal{K}(\mathscr{W})$ *is nuclear if and only if* $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N} \; : \; w_n / w_m \in L^1$

- This follows from: Vogt's sequence space representation and characterization of nuclear Köthe sequence spaces.
- Gelfand-Shilov showed necessity under stronger conditions.
- Kruse has (2020) studied the problem on open $\Omega \subset \mathbb{R}^d$.

- Consider $\mathscr{W} = (w_n)_{n\in\mathbb{N}}$ with $w_n \in C(\mathbb{R}^d)$ and $1 \leq w_1 \leq w_2 \leq \cdots$.
- $\mathcal{K}(\mathscr{W}) = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \max_{|\alpha|\leq n} \|\varphi^{(\alpha)} w_n\|_{L^\infty} < \infty \quad \forall n \in \mathbb{N}\}$.

Theorem

Assume the mild regularity hypothesis: $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N}\, \exists C > 0$

$$\sup_{|y|\leq 1} w_n(x+y) \leq C w_m(x), \quad \forall x \in \mathbb{R}^d.$$

Then, $\mathcal{K}(\mathscr{W})$ *is nuclear if and only if* $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N} \, : \, w_n/w_m \in L^1$

- This follows from: Vogt's sequence space representation and characterization of nuclear Köthe sequence spaces.
- Gelfand-Shilov showed necessity under stronger conditions.
- Kruse has (2020) studied the problem on open $\Omega \subset \mathbb{R}^d$.

- Consider $\mathscr{W} = (w_n)_{n\in\mathbb{N}}$ with $w_n \in C(\mathbb{R}^d)$ and $1 \leq w_1 \leq w_2 \leq \cdots$.
- $\mathcal{K}(\mathscr{W}) = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \max_{|\alpha|\leq n} \|\varphi^{(\alpha)} w_n\|_{L^\infty} < \infty \quad \forall n \in \mathbb{N}\}$.

Theorem

Assume the mild regularity hypothesis: $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N}\, \exists C > 0$

$$\sup_{|y|\leq 1} w_n(x+y) \leq C w_m(x), \quad \forall x \in \mathbb{R}^d.$$

Then, $\mathcal{K}(\mathscr{W})$ *is nuclear if and only if* $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N} \;:\; w_n/w_m \in L^1$

- This follows from: Vogt's sequence space representation and characterization of nuclear Köthe sequence spaces.
- Gelfand-Shilov showed necessity under stronger conditions.
- Kruse has (2020) studied the problem on open $\Omega \subset \mathbb{R}^d$.

- Consider $\mathscr{W} = (w_n)_{n\in\mathbb{N}}$ with $w_n \in C(\mathbb{R}^d)$ and $1 \leq w_1 \leq w_2 \leq \cdots$.
- $\mathcal{K}(\mathscr{W}) = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \max_{|\alpha|\leq n} \|\varphi^{(\alpha)} w_n\|_{L^\infty} < \infty \quad \forall n \in \mathbb{N}\}$.

Theorem

Assume the mild regularity hypothesis: $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N}\, \exists C > 0$

$$\sup_{|y|\leq 1} w_n(x+y) \leq C w_m(x), \quad \forall x \in \mathbb{R}^d.$$

Then, $\mathcal{K}(\mathscr{W})$ *is nuclear if and only if* $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N} \ : \ w_n/w_m \in L^1$

- This follows from: Vogt's sequence space representation and characterization of nuclear Köthe sequence spaces.
- Gelfand-Shilov showed necessity under stronger conditions.
- Kruse has (2020) studied the problem on open $\Omega \subset \mathbb{R}^d$.

- Consider $\mathscr{W} = (w_n)_{n\in\mathbb{N}}$ with $w_n \in C(\mathbb{R}^d)$ and $1 \leq w_1 \leq w_2 \leq \cdots$.
- $\mathcal{K}(\mathscr{W}) = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \max_{|\alpha|\leq n} \|\varphi^{(\alpha)} w_n\|_{L^\infty} < \infty \quad \forall n \in \mathbb{N}\}$.

Theorem

Assume the mild regularity hypothesis: $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N}\, \exists C > 0$

$$\sup_{|y|\leq 1} w_n(x+y) \leq C w_m(x), \quad \forall x \in \mathbb{R}^d.$$

Then, $\mathcal{K}(\mathscr{W})$ *is nuclear if and only if* $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N} \,:\, w_n/w_m \in L^1$

- This follows from: Vogt's sequence space representation and characterization of nuclear Köthe sequence spaces.
- Gelfand-Shilov showed necessity under stronger conditions.
- Kruse has (2020) studied the problem on open $\Omega \subset \mathbb{R}^d$.

- Consider $\mathscr{W} = (w_n)_{n\in\mathbb{N}}$ with $w_n \in C(\mathbb{R}^d)$ and $1 \leq w_1 \leq w_2 \leq \cdots$.
- $\mathcal{K}(\mathscr{W}) = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \max_{|\alpha|\leq n} \|\varphi^{(\alpha)} w_n\|_{L^\infty} < \infty \quad \forall n \in \mathbb{N}\}$.

Theorem

Assume the mild regularity hypothesis: $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N}\, \exists C > 0$

$$\sup_{|y|\leq 1} w_n(x+y) \leq C w_m(x), \quad \forall x \in \mathbb{R}^d.$$

Then, $\mathcal{K}(\mathscr{W})$ *is nuclear if and only if* $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N} \ : \ w_n/w_m \in L^1$

- This follows from: Vogt's sequence space representation and characterization of nuclear Köthe sequence spaces.
- Gelfand-Shilov showed necessity under stronger conditions.
- Kruse has (2020) studied the problem on open $\Omega \subset \mathbb{R}^d$.

Nuclearity for Gelfand-Shilov spaces of ultradifferentiable functions

There has been recent interest in nuclearity and kernel theorems for Gelfand-Shilov spaces of type $\mathcal{S}$

- Pilipović, Prangoski, and myself (2018).
- Boiti, Jornet, Oliaro, and Schindl (2020, 2021).
- ...

However:

- Sufficient conditions in those works already contained in classical work by Mityagin (1960), up to minor modifications.
- Considered classes not stable under tensor products.

Mityagin results actually apply for general classes of Gelfand-Shilov spaces defined by weight matrices.

Our approach:

- Stable under tensor products, leading to new kernel theorems.
- Conterparts of $\mathcal{K}(\mathscr{W})$ in the ultradifferentiable context.

Nuclearity for Gelfand-Shilov spaces of ultradifferentiable functions

There has been recent interest in nuclearity and kernel theorems for Gelfand-Shilov spaces of type $\mathcal{S}$

- Pilipović, Prangoski, and myself (2018).
- Boiti, Jornet, Oliaro, and Schindl (2020, 2021).
- ...

However:

- Sufficient conditions in those works already contained in classical work by Mityagin (1960), up to minor modifications.
- Considered classes not stable under tensor products.

Mityagin results actually apply for general classes of Gelfand-Shilov spaces defined by weight matrices.

Our approach:

- Stable under tensor products, leading to new kernel theorems.
- Conterparts of $\mathcal{K}(\mathscr{W})$ in the ultradifferentiable context.

Nuclearity for Gelfand-Shilov spaces of ultradifferentiable functions

There has been recent interest in nuclearity and kernel theorems for Gelfand-Shilov spaces of type $\mathcal{S}$

- Pilipović, Prangoski, and myself (2018).
- Boiti, Jornet, Oliaro, and Schindl (2020, 2021).
- ...

However:

- Sufficient conditions in those works already contained in classical work by Mityagin (1960), up to minor modifications.
- Considered classes not stable under tensor products.

Mityagin results actually apply for general classes of Gelfand-Shilov spaces defined by weight matrices.

Our approach:

- Stable under tensor products, leading to new kernel theorems.
- Conterparts of $\mathcal{K}(\mathscr{W})$ in the ultradifferentiable context.

Nuclearity for Gelfand-Shilov spaces of ultradifferentiable functions

There has been recent interest in nuclearity and kernel theorems for Gelfand-Shilov spaces of type $\mathcal{S}$

- Pilipović, Prangoski, and myself (2018).
- Boiti, Jornet, Oliaro, and Schindl (2020, 2021).
- ...

However:

- Sufficient conditions in those works already contained in classical work by Mityagin (1960), up to minor modifications.
- Considered classes not stable under tensor products.

Mityagin results actually apply for general classes of Gelfand-Shilov spaces defined by weight matrices.

Our approach:

- Stable under tensor products, leading to new kernel theorems.
- Conterparts of $\mathcal{K}(\mathscr{W})$ in the ultradifferentiable context.

Nuclearity for Gelfand-Shilov spaces of ultradifferentiable functions

There has been recent interest in nuclearity and kernel theorems for Gelfand-Shilov spaces of type $\mathcal{S}$

- Pilipović, Prangoski, and myself (2018).
- Boiti, Jornet, Oliaro, and Schindl (2020, 2021).
- ...

However:

- Sufficient conditions in those works already contained in classical work by Mityagin (1960), up to minor modifications.
- Considered classes not stable under tensor products.

Mityagin results actually apply for general classes of Gelfand-Shilov spaces defined by weight matrices.

Our approach:

- Stable under tensor products, leading to new kernel theorems.
- Conterparts of $\mathcal{K}(\mathscr{W})$ in the ultradifferentiable context.

Nuclearity for Gelfand-Shilov spaces of ultradifferentiable functions

There has been recent interest in nuclearity and kernel theorems for Gelfand-Shilov spaces of type $\mathcal{S}$

- Pilipović, Prangoski, and myself (2018).
- Boiti, Jornet, Oliaro, and Schindl (2020, 2021).
- ...

However:

- Sufficient conditions in those works already contained in classical work by Mityagin (1960), up to minor modifications.
- Considered classes not stable under tensor products.

Mityagin results actually apply for general classes of Gelfand-Shilov spaces defined by weight matrices.

Our approach:

- Stable under tensor products, leading to new kernel theorems.
- Conterparts of $\mathcal{K}(\mathscr{W})$ in the ultradifferentiable context.

Gelfand-Shilov spaces: definition

A weight sequence $M = (M_\alpha)_\alpha$ is a (multi-)sequence of positive numbers such that $\lim_{\alpha\to\infty} M_\alpha^{1/|\alpha|} = \infty$ and $M_{\alpha+e_j}^2 \leq M_\alpha M_{\alpha+2e_j}$, $\forall \alpha \in \mathbb{N}^d$.

A weight sequence system $\mathfrak{M} = \{M^\lambda : \lambda \in \mathbb{R}_+\}$ is a family of weight sequences such that $M^\lambda \leq M^\mu$ when $\lambda \leq \mu$.

A family $\mathscr{W} = \{w^\lambda : \lambda \in \mathbb{R}_+\}$ of positive continuous functions is called a weight function system if $1 \leq w^\lambda \leq w^\mu$ when $\mu \leq \lambda$.

Given $\lambda > 0$, we consider the Banach spaces

$$\mathcal{S}^{M^\lambda}_{w^\lambda} = \{\varphi \in C^\infty(\mathbb{R}^d) : \|\varphi\|_{\mathcal{S}^{M^\lambda}_{w^\lambda}} = \sup_{\alpha\in\mathbb{N}^d} \frac{1}{M^\lambda_\alpha} \|\varphi^{(\alpha)} w^\lambda\|_{L^\infty(\mathbb{R}^d)} < \infty\},$$

General Gelfand-Shilov spaces of Beurling and Roumieu type

$$\mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})} = \varprojlim_{\lambda\to 0^+} \mathcal{S}^{M^\lambda}_{w^\lambda}, \qquad \mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}} = \varinjlim_{\lambda\to\infty} \mathcal{S}^{M^\lambda}_{w^\lambda}.$$

$\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ is the common notation for both the Beurling and Roumieu spaces.

Gelfand-Shilov spaces: definition

A weight sequence $M = (M_\alpha)_\alpha$ is a (multi-)sequence of positive numbers such that $\lim_{\alpha\to\infty} M_\alpha^{1/|\alpha|} = \infty$ and $M^2_{\alpha+e_j} \le M_\alpha M_{\alpha+2e_j}$, $\forall \alpha \in \mathbb{N}^d$.

A weight sequence system $\mathfrak{M} = \{M^\lambda : \lambda \in \mathbb{R}_+\}$ is a family of weight sequences such that $M^\lambda \le M^\mu$ when $\lambda \le \mu$.

A family $\mathscr{W} = \{w^\lambda : \lambda \in \mathbb{R}_+\}$ of positive continuous functions is called a weight function system if $1 \le w^\lambda \le w^\mu$ when $\mu \le \lambda$.

Given $\lambda > 0$, we consider the Banach spaces

$$\mathcal{S}^{M^\lambda}_{w^\lambda} = \{\varphi \in C^\infty(\mathbb{R}^d) : \|\varphi\|_{\mathcal{S}^{M^\lambda}_{w^\lambda}} = \sup_{\alpha\in\mathbb{N}^d} \frac{1}{M^\lambda_\alpha} \|\varphi^{(\alpha)} w^\lambda\|_{L^\infty(\mathbb{R}^d)} < \infty\},$$

General Gelfand-Shilov spaces of Beurling and Roumieu type

$$\mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})} = \varprojlim_{\lambda\to 0^+} \mathcal{S}^{M^\lambda}_{w^\lambda}, \qquad \mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}} = \varinjlim_{\lambda\to\infty} \mathcal{S}^{M^\lambda}_{w^\lambda}.$$

$\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ is the common notation for both the Beurling and Roumieu spaces.

Gelfand-Shilov spaces: definition

A weight sequence $M = (M_\alpha)_\alpha$ is a (multi-)sequence of positive numbers such that $\lim_{\alpha\to\infty} M_\alpha^{1/|\alpha|} = \infty$ and $M_{\alpha+e_j}^2 \leq M_\alpha M_{\alpha+2e_j}$, $\forall \alpha \in \mathbb{N}^d$.

A weight sequence system $\mathfrak{M} = \{M^\lambda : \lambda \in \mathbb{R}_+\}$ is a family of weight sequences such that $M^\lambda \leq M^\mu$ when $\lambda \leq \mu$.

A family $\mathscr{W} = \{w^\lambda : \lambda \in \mathbb{R}_+\}$ of positive continuous functions is called a weight function system if $1 \leq w^\lambda \leq w^\mu$ when $\mu \leq \lambda$.

Given $\lambda > 0$, we consider the Banach spaces

$$\mathcal{S}^{M^\lambda}_{w^\lambda} = \{\varphi \in C^\infty(\mathbb{R}^d) : \|\varphi\|_{\mathcal{S}^{M^\lambda}_{w^\lambda}} = \sup_{\alpha\in\mathbb{N}^d} \frac{1}{M^\lambda_\alpha} \|\varphi^{(\alpha)} w^\lambda\|_{L^\infty(\mathbb{R}^d)} < \infty\},$$

General Gelfand-Shilov spaces of Beurling and Roumieu type

$$\mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})} = \varprojlim_{\lambda\to 0^+} \mathcal{S}^{M^\lambda}_{w^\lambda}, \qquad \mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}} = \varinjlim_{\lambda\to\infty} \mathcal{S}^{M^\lambda}_{w^\lambda}.$$

$\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ is the common notation for both the Beurling and Roumieu spaces.

Gelfand-Shilov spaces: definition

A weight sequence $M = (M_\alpha)_\alpha$ is a (multi-)sequence of positive numbers such that $\lim_{\alpha\to\infty} M_\alpha^{1/|\alpha|} = \infty$ and $M_{\alpha+e_j}^2 \le M_\alpha M_{\alpha+2e_j}$, $\forall \alpha \in \mathbb{N}^d$.

A weight sequence system $\mathfrak{M} = \{M^\lambda : \lambda \in \mathbb{R}_+\}$ is a family of weight sequences such that $M^\lambda \le M^\mu$ when $\lambda \le \mu$.

A family $\mathscr{W} = \{w^\lambda : \lambda \in \mathbb{R}_+\}$ of positive continuous functions is called a weight function system if $1 \le w^\lambda \le w^\mu$ when $\mu \le \lambda$.

Given $\lambda > 0$, we consider the Banach spaces

$$\mathcal{S}^{M^\lambda}_{w^\lambda} = \{\varphi \in C^\infty(\mathbb{R}^d) : \|\varphi\|_{\mathcal{S}^{M^\lambda}_{w^\lambda}} = \sup_{\alpha\in\mathbb{N}^d} \frac{1}{M^\lambda_\alpha} \|\varphi^{(\alpha)} w^\lambda\|_{L^\infty(\mathbb{R}^d)} < \infty\},$$

General Gelfand-Shilov spaces of Beurling and Roumieu type

$$\mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})} = \varprojlim_{\lambda\to0^+} \mathcal{S}^{M^\lambda}_{w^\lambda}, \qquad \mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}} = \varinjlim_{\lambda\to\infty} \mathcal{S}^{M^\lambda}_{w^\lambda}.$$

$\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ is the common notation for both the Beurling and Roumieu spaces.

Gelfand-Shilov spaces: definition

A weight sequence $M = (M_\alpha)_\alpha$ is a (multi-)sequence of positive numbers such that $\lim_{\alpha\to\infty} M_\alpha^{1/|\alpha|} = \infty$ and $M_{\alpha+e_j}^2 \le M_\alpha M_{\alpha+2e_j}$, $\forall \alpha \in \mathbb{N}^d$.

A weight sequence system $\mathfrak{M} = \{M^\lambda : \lambda \in \mathbb{R}_+\}$ is a family of weight sequences such that $M^\lambda \le M^\mu$ when $\lambda \le \mu$.

A family $\mathscr{W} = \{w^\lambda : \lambda \in \mathbb{R}_+\}$ of positive continuous functions is called a weight function system if $1 \le w^\lambda \le w^\mu$ when $\mu \le \lambda$.

Given $\lambda > 0$, we consider the Banach spaces

$$\mathcal{S}_{w^\lambda}^{M^\lambda} = \Big\{\varphi \in C^\infty(\mathbb{R}^d) : \|\varphi\|_{\mathcal{S}_{w^\lambda}^{M^\lambda}} = \sup_{\alpha\in\mathbb{N}^d} \frac{1}{M_\alpha^\lambda}\|\varphi^{(\alpha)} w^\lambda\|_{L^\infty(\mathbb{R}^d)} < \infty\Big\},$$

General Gelfand-Shilov spaces of Beurling and Roumieu type

$$\mathcal{S}_{(\mathscr{W})}^{(\mathfrak{M})} = \varprojlim_{\lambda\to 0^+} \mathcal{S}_{w^\lambda}^{M^\lambda}, \qquad \mathcal{S}_{\{\mathscr{W}\}}^{\{\mathfrak{M}\}} = \varinjlim_{\lambda\to\infty} \mathcal{S}_{w^\lambda}^{M^\lambda}.$$

$\mathcal{S}_{[\mathscr{W}]}^{[\mathfrak{M}]}$ is the common notation for both the Beurling and Roumieu spaces.

Examples of weight sequence and function systems

One can generate important examples of weight systems as follows

- Via a weight sequence $M = (M_\alpha)_{\alpha \in \mathbb{N}^d}$:

$$\mathfrak{M}_M = \{(\lambda^{|\alpha|} M_\alpha)_{\alpha \in \mathbb{N}^d} : \lambda \in \mathbb{R}_+\}, \qquad \mathscr{W}_M = \{\exp \omega_M(\cdot/\lambda) : \lambda \in \mathbb{R}_+\}$$

where $\omega_M(x) = \sup_{\alpha \in \mathbb{N}^d} \log \frac{|x^\alpha| M_0}{M_\alpha}$, $x \in \mathbb{R}^d$.

- Via a single non-decreasing weight function $\omega : [0, \infty) \to [0, \infty)$

$$\mathscr{W}_\omega = \{\exp(\frac{1}{\lambda}\omega(|\cdot|)) : \lambda \in \mathbb{R}_+\} \qquad \text{(Beurling-Björck type)}$$

- If additionally ω is a BMT weight function, i.e., $\omega(2t) = O(\omega(t))$, $\log t = o(\omega(t))$, and $\omega \circ \exp$ is convex,

$$\mathfrak{M}_\omega = \{(\exp(\frac{1}{\lambda}\phi^*(\lambda|\alpha|)))_{\alpha \in \mathbb{N}^d} : \lambda \in \mathbb{R}_+\},$$

with $\phi^*(y) = \sup_{x \geq 0}(xy - \omega(e^x))$ the Young conjugate of $\omega \circ \exp$.

Classical spaces

$$\mathcal{S}^{[M]}_{[A]} := \mathcal{S}^{[\mathfrak{M}_M]}_{[\mathscr{W}_A]} \text{ (Gelfand-Shilov)} \quad \text{and} \quad \mathcal{S}^{[\omega]}_{[\eta]} := \mathcal{S}^{[\mathfrak{M}_\omega]}_{[\mathscr{W}_\eta]} \text{ (Beurling-Björck)}$$

Examples of weight sequence and function systems

One can generate important examples of weight systems as follows

- Via a weight sequence $M = (M_\alpha)_{\alpha\in\mathbb{N}^d}$:

$$\mathfrak{M}_M = \{(\lambda^{|\alpha|} M_\alpha)_{\alpha\in\mathbb{N}^d} : \lambda \in \mathbb{R}_+\}, \qquad \mathscr{W}_M = \{\exp \omega_M(\cdot/\lambda) : \lambda \in \mathbb{R}_+\}$$

where $\omega_M(x) = \sup_{\alpha\in\mathbb{N}^d} \log \frac{|x^\alpha| M_0}{M_\alpha}$, $x \in \mathbb{R}^d$.

- Via a single non-decreasing weight function $\omega : [0,\infty) \to [0,\infty)$

$$\mathscr{W}_\omega = \{\exp(\frac{1}{\lambda}\omega(|\cdot|)) : \lambda \in \mathbb{R}_+\} \qquad \text{(Beurling-Björck type)}$$

- If additionally ω is a BMT weight function, i.e., $\omega(2t) = O(\omega(t))$, $\log t = o(\omega(t))$, and $\omega \circ \exp$ is convex,

$$\mathfrak{M}_\omega = \{(\exp(\frac{1}{\lambda}\phi^*(\lambda|\alpha|)))_{\alpha\in\mathbb{N}^d} : \lambda \in \mathbb{R}_+\},$$

with $\phi^*(y) = \sup_{x\geq 0}(xy - \omega(e^x))$ the Young conjugate of $\omega \circ \exp$.

Classical spaces

$$\mathcal{S}^{[M]}_{[A]} := \mathcal{S}^{[\mathfrak{M}_M]}_{[\mathscr{W}_A]} \text{ (Gelfand-Shilov)} \quad \text{and} \quad \mathcal{S}^{[\omega]}_{[\eta]} := \mathcal{S}^{[\mathfrak{M}_\omega]}_{[\mathscr{W}_\eta]} \text{ (Beurling-Björck)}$$

Examples of weight sequence and function systems

One can generate important examples of weight systems as follows

- Via a weight sequence $M = (M_\alpha)_{\alpha\in\mathbb{N}^d}$:

$$\mathfrak{M}_M = \{(\lambda^{|\alpha|} M_\alpha)_{\alpha\in\mathbb{N}^d} : \lambda \in \mathbb{R}_+\}, \qquad \mathscr{W}_M = \{\exp \omega_M(\cdot/\lambda) : \lambda \in \mathbb{R}_+\}$$

where $\omega_M(x) = \sup_{\alpha\in\mathbb{N}^d} \log \dfrac{|x^\alpha| M_0}{M_\alpha}$, $x \in \mathbb{R}^d$.

- Via a single non-decreasing weight function $\omega : [0,\infty) \to [0,\infty)$

$$\mathscr{W}_\omega = \{\exp(\frac{1}{\lambda}\omega(|\cdot|)) : \lambda \in \mathbb{R}_+\} \qquad \text{(Beurling-Björck type)}$$

- If additionally ω is a BMT weight function, i.e., $\omega(2t) = O(\omega(t))$, $\log t = o(\omega(t))$, and $\omega \circ \exp$ is convex,

$$\mathfrak{M}_\omega = \{(\exp(\frac{1}{\lambda}\phi^*(\lambda|\alpha|)))_{\alpha\in\mathbb{N}^d} : \lambda \in \mathbb{R}_+\},$$

with $\phi^*(y) = \sup_{x\geq 0}(xy - \omega(e^x))$ the Young conjugate of $\omega \circ \exp$.

Classical spaces

$$\mathcal{S}^{[M]}_{[A]} := \mathcal{S}^{[\mathfrak{M}_M]}_{[\mathscr{W}_A]} \text{ (Gelfand-Shilov)} \quad \text{and} \quad \mathcal{S}^{[\omega]}_{[\eta]} := \mathcal{S}^{[\mathfrak{M}_\omega]}_{[\mathscr{W}_\eta]} \text{ (Beurling-Björck)}$$

Examples of weight sequence and function systems

One can generate important examples of weight systems as follows

- Via a weight sequence $M = (M_\alpha)_{\alpha\in\mathbb{N}^d}$:

$$\mathfrak{M}_M = \{(\lambda^{|\alpha|} M_\alpha)_{\alpha\in\mathbb{N}^d} : \lambda \in \mathbb{R}_+\}, \qquad \mathscr{W}_M = \{\exp \omega_M(\cdot/\lambda) : \lambda \in \mathbb{R}_+\}$$

  where $\omega_M(x) = \sup_{\alpha\in\mathbb{N}^d} \log \dfrac{|x^\alpha| M_0}{M_\alpha}$, $x \in \mathbb{R}^d$.

- Via a single non-decreasing weight function $\omega : [0,\infty) \to [0,\infty)$

$$\mathscr{W}_\omega = \{\exp(\frac{1}{\lambda}\omega(|\cdot|)) : \lambda \in \mathbb{R}_+\} \qquad \text{(Beurling-Björck type)}$$

- If additionally ω is a BMT weight function, i.e., $\omega(2t) = O(\omega(t))$, $\log t = o(\omega(t))$, and $\omega \circ \exp$ is convex,

$$\mathfrak{M}_\omega = \{(\exp(\frac{1}{\lambda}\phi^*(\lambda|\alpha|)))_{\alpha\in\mathbb{N}^d} : \lambda \in \mathbb{R}_+\},$$

  with $\phi^*(y) = \sup_{x\geq 0}(xy - \omega(e^x))$ the Young conjugate of $\omega \circ \exp$.

Classical spaces

$$\mathcal{S}^{[M]}_{[A]} := \mathcal{S}^{[\mathfrak{M}_M]}_{[\mathscr{W}_A]} \text{ (Gelfand-Shilov)} \quad \text{and} \quad \mathcal{S}^{[\omega]}_{[\eta]} := \mathcal{S}^{[\mathfrak{M}_\omega]}_{[\mathscr{W}_\eta]} \text{ (Beurling-Björck)}$$

Notation

We employ $[\cdot]$ as a common notation for the Beurling and Roumieu cases. The conditions below should be preceded by the quantifiers:

Beurling case: $\forall \lambda \in \mathbb{R}_+ \ \exists \mu \in \mathbb{R}_+$;

Roumieu case: $\forall \mu \in \mathbb{R}_+ \ \exists \lambda \in \mathbb{R}_+$.

- We consider the following conditions on $\mathfrak{M}$:

  [L] $\quad \forall L > 0 : L^{|\alpha|} M^{\mu}_{\alpha} \leq C M^{\lambda}_{\alpha}$;

  $[\mathfrak{M}.2]'$ $\quad \exists H > 0 : M^{\mu}_{\alpha + e_j} \leq C H^{|\alpha|} M^{\lambda}_{\alpha}$.

- We also consider the following conditions on $\mathscr{W}$:

  [wM] $\quad \sup_{|y| \leq 1} w^{\lambda}(x+y) \leq C w^{\mu}(x)$

  [M] $\quad w^{\lambda}(x+y) \leq C w^{\mu}(x) w^{\mu}(y)$

  [N] $\quad w^{\lambda}/w^{\mu} \in L^1(\mathbb{R}^d)$

Notation

We employ $[\cdot]$ as a common notation for the Beurling and Roumieu cases. The conditions below should be preceded by the quantifiers:

Beurling case: $\forall \lambda \in \mathbb{R}_+ \ \exists \mu \in \mathbb{R}_+$;

Roumieu case: $\forall \mu \in \mathbb{R}_+ \ \exists \lambda \in \mathbb{R}_+$.

- We consider the following conditions on $\mathfrak{M}$:

  [L] $\quad \forall L > 0 : L^{|\alpha|} M^{\mu}_{\alpha} \leq C M^{\lambda}_{\alpha}$;

  $[\mathfrak{M}.2]'$ $\quad \exists H > 0 : M^{\mu}_{\alpha + e_j} \leq C H^{|\alpha|} M^{\lambda}_{\alpha}$.

- We also consider the following conditions on $\mathscr{W}$:

  [wM] $\quad \sup_{|y| \leq 1} w^{\lambda}(x+y) \leq C w^{\mu}(x)$

  [M] $\quad w^{\lambda}(x+y) \leq C w^{\mu}(x) w^{\mu}(y)$

  [N] $\quad w^{\lambda}/w^{\mu} \in L^1(\mathbb{R}^d)$

Conditions on weight systems related to nuclearity

Notation

We employ $[\cdot]$ as a common notation for the Beurling and Roumieu cases. The conditions below should be preceded by the quantifiers:

Beurling case: $\forall \lambda \in \mathbb{R}_+ \ \exists \mu \in \mathbb{R}_+$;

Roumieu case: $\forall \mu \in \mathbb{R}_+ \ \exists \lambda \in \mathbb{R}_+$.

- We consider the following conditions on $\mathfrak{M}$:

$$[\mathrm{L}] \qquad \forall L > 0 : L^{|\alpha|} M^{\mu}_{\alpha} \leq C M^{\lambda}_{\alpha};$$
$$[\mathfrak{M}.2]' \qquad \exists H > 0 : M^{\mu}_{\alpha+e_j} \leq C H^{|\alpha|} M^{\lambda}_{\alpha}.$$

- We also consider the following conditions on $\mathscr{W}$:

$$[\mathrm{wM}] \qquad \sup_{|y|\leq 1} w^{\lambda}(x+y) \leq C w^{\mu}(x)$$
$$[\mathrm{M}] \qquad w^{\lambda}(x+y) \leq C w^{\mu}(x) w^{\mu}(y)$$
$$[\mathrm{N}] \qquad w^{\lambda}/w^{\mu} \in L^1(\mathbb{R}^d)$$

We have the following general sufficient conditions

Theorem (Debrouwere, Neyt, and V., 2021)

- Let $\mathfrak{M}$ satisfy [L] and $[\mathfrak{M}.2]'$.
- Let $\mathscr{W}$ satisfy [wM] and [N].

Then, $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ is nuclear.

A converse:

Theorem (Debrouwere, Neyt, and V., 2021)

Let $\mathfrak{M}$ satisfy [L], let $\mathscr{W}$ satisfy [M]. If $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}(\mathbb{R}^d)$ is non-trivial and nuclear, then $\mathscr{W}$ satisfies [N].

We have the following general sufficient conditions

Theorem (Debrouwere, Neyt, and V., 2021)

- Let $\mathfrak{M}$ satisfy [L] and $[\mathfrak{M}.2]'$.
- Let $\mathscr{W}$ satisfy [wM] and [N].

Then, $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ is nuclear.

A converse:

Theorem (Debrouwere, Neyt, and V., 2021)

Let $\mathfrak{M}$ satisfy [L], let $\mathscr{W}$ satisfy [M]. If $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}(\mathbb{R}^d)$ is non-trivial and nuclear, then $\mathscr{W}$ satisfies [N].

We have the following general sufficient conditions

Theorem (Debrouwere, Neyt, and V., 2021)

- Let $\mathfrak{M}$ satisfy [L] and $[\mathfrak{M}.2]'$.
- Let $\mathscr{W}$ satisfy [wM] and [N].

Then, $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ is nuclear.

A converse:

Theorem (Debrouwere, Neyt, and V., 2021)

Let $\mathfrak{M}$ satisfy [L], let $\mathscr{W}$ satisfy [M]. If $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}(\mathbb{R}^d)$ is non-trivial and nuclear, then $\mathscr{W}$ satisfies [N].

We have the following general sufficient conditions

Theorem (Debrouwere, Neyt, and V., 2021)

- Let $\mathfrak{M}$ satisfy [L] and $[\mathfrak{M}.2]'$.
- Let $\mathscr{W}$ satisfy [wM] and [N].

Then, $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ is nuclear.

A converse:

Theorem (Debrouwere, Neyt, and V., 2021)

Let $\mathfrak{M}$ satisfy [L], let $\mathscr{W}$ satisfy [M]. If $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}(\mathbb{R}^d)$ is non-trivial and nuclear, then $\mathscr{W}$ satisfies [N].

The necessity of $[\mathfrak{M}.2]'$ for nuclearity

A weigh sequence system $\mathfrak{M} = \{M^\lambda : \lambda \in \mathbb{R}_+\}$ is called

- accelerating if $M^\lambda_{\alpha+e_j}/M^\lambda_\alpha \leq M^\mu_{\alpha+e_j}/M^\mu_\alpha$ when $\lambda \leq \mu$.
- isotropic if $M^\lambda_\alpha = M^\lambda_\beta$ whenever $|\alpha| = |\beta|$.
- $\mathfrak{M}$ is called isotropically decomposable if, after some permutation of the multi-indices, it can be written as a tensor product of isotropic weight sequence systems.

Theorem (Debrouwere, Neyt, and V., 2021)

Suppose that:

- $\mathfrak{M}$ satisfies [L] and is isotropically decomposable and accelerating.
- $\mathscr{W}$ be a weight function system satisfying [M].

If $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]} \neq \{0\}$, the following are equivalent:

1. $\mathfrak{M}$ satisfies $[\mathfrak{M}.2]'$ and $\mathscr{W}$ satisfies [N].
2. The space $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ is nuclear.

The necessity of $[\mathfrak{M}.2]'$ for nuclearity

A weigh sequence system $\mathfrak{M} = \{M^\lambda : \lambda \in \mathbb{R}_+\}$ is called

- accelerating if $M^\lambda_{\alpha+e_j}/M^\lambda_\alpha \leq M^\mu_{\alpha+e_j}/M^\mu_\alpha$ when $\lambda \leq \mu$.
- isotropic if $M^\lambda_\alpha = M^\lambda_\beta$ whenever $|\alpha| = |\beta|$.
- $\mathfrak{M}$ is called isotropically decomposable if, after some permutation of the multi-indices, it can be written as a tensor product of isotropic weight sequence systems.

Theorem (Debrouwere, Neyt, and V., 2021)

Suppose that:

- $\mathfrak{M}$ satisfies [L] and is isotropically decomposable and accelerating.
- $\mathscr{W}$ be a weight function system satisfying [M].

If $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]} \neq \{0\}$, the following are equivalent:

1. $\mathfrak{M}$ satisfies $[\mathfrak{M}.2]'$ and $\mathscr{W}$ satisfies [N].
2. The space $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ is nuclear.

The necessity of $[\mathfrak{M}.2]'$ for nuclearity

A weigh sequence system $\mathfrak{M} = \{M^\lambda : \lambda \in \mathbb{R}_+\}$ is called

- accelerating if $M^\lambda_{\alpha+e_j}/M^\lambda_\alpha \leq M^\mu_{\alpha+e_j}/M^\mu_\alpha$ when $\lambda \leq \mu$.
- isotropic if $M^\lambda_\alpha = M^\lambda_\beta$ whenever $|\alpha| = |\beta|$.
- $\mathfrak{M}$ is called isotropically decomposable if, after some permutation of the multi-indices, it can be written as a tensor product of isotropic weight sequence systems.

Theorem (Debrouwere, Neyt, and V., 2021)

Suppose that:

- $\mathfrak{M}$ satisfies [L] and is isotropically decomposable and accelerating.
- $\mathscr{W}$ be a weight function system satisfying [M].

If $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]} \neq \{0\}$, the following are equivalent:

1. $\mathfrak{M}$ satisfies $[\mathfrak{M}.2]'$ and $\mathscr{W}$ satisfies [N].
2. The space $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ is nuclear.

The necessity of $[\mathfrak{M}.2]'$ for nuclearity

A weigh sequence system $\mathfrak{M} = \{M^\lambda : \ \lambda \in \mathbb{R}_+\}$ is called

- accelerating if $M^\lambda_{\alpha+e_j}/M^\lambda_\alpha \leq M^\mu_{\alpha+e_j}/M^\mu_\alpha$ when $\lambda \leq \mu$.
- isotropic if $M^\lambda_\alpha = M^\lambda_\beta$ whenever $|\alpha| = |\beta|$.
- $\mathfrak{M}$ is called isotropically decomposable if, after some permutation of the multi-indices, it can be written as a tensor product of isotropic weight sequence systems.

Theorem (Debrouwere, Neyt, and V., 2021)

Suppose that:

- $\mathfrak{M}$ satisfies [L] and is isotropically decomposable and accelerating.
- $\mathscr{W}$ be a weight function system satisfying [M].

If $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]} \neq \{0\}$, the following are equivalent:

1. $\mathfrak{M}$ satisfies $[\mathfrak{M}.2]'$ and $\mathscr{W}$ satisfies [N].
2. The space $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ is nuclear.

The necessity of $[\mathfrak{M}.2]'$ for nuclearity

A weigh sequence system $\mathfrak{M} = \{M^\lambda : \lambda \in \mathbb{R}_+\}$ is called

- accelerating if $M^\lambda_{\alpha+e_j}/M^\lambda_\alpha \leq M^\mu_{\alpha+e_j}/M^\mu_\alpha$ when $\lambda \leq \mu$.
- isotropic if $M^\lambda_\alpha = M^\lambda_\beta$ whenever $|\alpha| = |\beta|$.
- $\mathfrak{M}$ is called isotropically decomposable if, after some permutation of the multi-indices, it can be written as a tensor product of isotropic weight sequence systems.

Theorem (Debrouwere, Neyt, and V., 2021)

Suppose that:

- $\mathfrak{M}$ satisfies [L] and is isotropically decomposable and accelerating.
- $\mathscr{W}$ be a weight function system satisfying [M].

If $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]} \neq \{0\}$, the following are equivalent:

1. $\mathfrak{M}$ satisfies $[\mathfrak{M}.2]'$ and $\mathscr{W}$ satisfies [N].
2. The space $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ is nuclear.

Special cases

Suppose the space under consideration is non-trivial.

1. Let M and A be isotropic weight sequences.

Corollary

$\mathcal{S}^{[M]}_{[A]}$ *is nuclear if and only if both* M *and* A *satisfy* $(M.2)'$

2. Let ω be a BMT weight function and let $\eta : [0,\infty) \to [0,\infty)$ be non-decreasing and satisfy $\eta(2t) = O(\eta(t))$.

Corollary

$\mathcal{S}^{[\omega]}_{[\eta]}$ *is nuclear if and only if* η *satisfies:*

Beurling case: $\log t = O(\eta(t))$; *Roumieu case:* $\log t = o(\eta(t))$.

A better result for $\mathcal{S}^{[\omega]}_{[\eta]}$ can be obtained, that is the subject of our paper:

Characterization of nuclearity for Beurling-Björck spaces, Proc. Amer. Math. Soc. 12 (2020), 5171–5180.

Special cases

Suppose the space under consideration is non-trivial.

1. Let M and A be isotropic weight sequences.

Corollary

$\mathcal{S}^{[M]}_{[A]}$ *is nuclear if and only if both M and A satisfy $(M.2)'$*

2. Let ω be a BMT weight function and let $\eta : [0,\infty) \to [0,\infty)$ be non-decreasing and satisfy $\eta(2t) = O(\eta(t))$.

Corollary

$\mathcal{S}^{[\omega]}_{[\eta]}$ *is nuclear if and only if η satisfies:*

Beurling case: $\log t = O(\eta(t))$; *Roumieu case:* $\log t = o(\eta(t))$.

A better result for $\mathcal{S}^{[\omega]}_{[\eta]}$ can be obtained, that is the subject of our paper:

Characterization of nuclearity for Beurling-Björck spaces, Proc. Amer. Math. Soc. 12 (2020), 5171–5180.

Special cases

Suppose the space under consideration is non-trivial.

1. Let M and A be isotropic weight sequences.

Corollary

$\mathcal{S}^{[M]}_{[A]}$ *is nuclear if and only if both M and A satisfy* $(M.2)'$

2. Let ω be a BMT weight function and let $\eta : [0, \infty) \to [0, \infty)$ be non-decreasing and satisfy $\eta(2t) = O(\eta(t))$.

Corollary

$\mathcal{S}^{[\omega]}_{[\eta]}$ *is nuclear if and only if η satisfies:*

Beurling case: $\log t = O(\eta(t))$; *Roumieu case:* $\log t = o(\eta(t))$.

A better result for $\mathcal{S}^{[\omega]}_{[\eta]}$ can be obtained, that is the subject of our paper:

Characterization of nuclearity for Beurling-Björck spaces, Proc. Amer. Math. Soc. 12 (2020), 5171–5180.

Application - Kernel theorems

Theorem (Debrouwere, Neyt, and V., 2021)

Assume $\mathfrak{M}_1$, $\mathfrak{M}_2$ satisfy [L] and $[\mathfrak{M}.2]'$ and $\mathscr{W}_1$, $\mathscr{W}_2$ satisfy [wM] and [N]. Then,

$$\mathcal{S}^{[\mathfrak{M}_1 \otimes \mathfrak{M}_2]}_{[\mathscr{W}_1 \otimes \mathscr{W}_2]}(\mathbb{R}^{d_1+d_2}) \cong \mathcal{S}^{[\mathfrak{M}_1]}_{[\mathscr{W}_1]}(\mathbb{R}^{d_1}) \widehat{\otimes} \mathcal{S}^{[\mathfrak{M}_2]}_{[\mathscr{W}_2]}(\mathbb{R}^{d_2})$$

$$\cong \mathcal{L}_b(\mathcal{S}^{[\mathfrak{M}_1]}_{[\mathscr{W}_1]}(\mathbb{R}^{d_1})', \mathcal{S}^{[\mathfrak{M}_2]}_{[\mathscr{W}_2]}(\mathbb{R}^{d_2})).$$

Consequently, we have the kernel theorem:

$$\boxed{\mathcal{S}^{[\mathfrak{M}_1 \otimes \mathfrak{M}_2]}_{[\mathscr{W}_1 \otimes \mathscr{W}_2]}(\mathbb{R}^{d_1+d_2})' \cong \mathcal{L}_b(\mathcal{S}^{[\mathfrak{M}_1]}_{[\mathscr{W}_1]}(\mathbb{R}^{d_1}), \mathcal{S}^{[\mathfrak{M}_2]}_{[\mathscr{W}_2]}(\mathbb{R}^{d_2})')}.$$

Theorem (Debrouwere, Neyt, and V., 2021)

Assume $\mathfrak{M}_1$, $\mathfrak{M}_2$ satisfy [L] and $[\mathfrak{M}.2]'$ and $\mathscr{W}_1$, $\mathscr{W}_2$ satisfy [wM] and [N]. Then,

$$\mathcal{S}^{[\mathfrak{M}_1 \otimes \mathfrak{M}_2]}_{[\mathscr{W}_1 \otimes \mathscr{W}_2]}(\mathbb{R}^{d_1+d_2}) \cong \mathcal{S}^{[\mathfrak{M}_1]}_{[\mathscr{W}_1]}(\mathbb{R}^{d_1}) \widehat{\otimes} \mathcal{S}^{[\mathfrak{M}_2]}_{[\mathscr{W}_2]}(\mathbb{R}^{d_2})$$

$$\cong \mathcal{L}_b(\mathcal{S}^{[\mathfrak{M}_1]}_{[\mathscr{W}_1]}(\mathbb{R}^{d_1})', \mathcal{S}^{[\mathfrak{M}_2]}_{[\mathscr{W}_2]}(\mathbb{R}^{d_2})).$$

Consequently, we have the kernel theorem:

$$\boxed{\mathcal{S}^{[\mathfrak{M}_1 \otimes \mathfrak{M}_2]}_{[\mathscr{W}_1 \otimes \mathscr{W}_2]}(\mathbb{R}^{d_1+d_2})' \cong \mathcal{L}_b(\mathcal{S}^{[\mathfrak{M}_1]}_{[\mathscr{W}_1]}(\mathbb{R}^{d_1}), \mathcal{S}^{[\mathfrak{M}_2]}_{[\mathscr{W}_2]}(\mathbb{R}^{d_2})')}.$$

Theorem (Debrouwere, Neyt, and V., 2021)

Assume $\mathfrak{M}_1$, $\mathfrak{M}_2$ satisfy [L] and $[\mathfrak{M}.2]'$ and $\mathscr{W}_1$, $\mathscr{W}_2$ satisfy [wM] and [N]. Then,

$$\mathcal{S}^{[\mathfrak{M}_1 \otimes \mathfrak{M}_2]}_{[\mathscr{W}_1 \otimes \mathscr{W}_2]}(\mathbb{R}^{d_1+d_2}) \cong \mathcal{S}^{[\mathfrak{M}_1]}_{[\mathscr{W}_1]}(\mathbb{R}^{d_1}) \widehat{\otimes} \mathcal{S}^{[\mathfrak{M}_2]}_{[\mathscr{W}_2]}(\mathbb{R}^{d_2})$$

$$\cong \mathcal{L}_b(\mathcal{S}^{[\mathfrak{M}_1]}_{[\mathscr{W}_1]}(\mathbb{R}^{d_1})', \mathcal{S}^{[\mathfrak{M}_2]}_{[\mathscr{W}_2]}(\mathbb{R}^{d_2})).$$

Consequently, we have the kernel theorem:

$$\boxed{\mathcal{S}^{[\mathfrak{M}_1 \otimes \mathfrak{M}_2]}_{[\mathscr{W}_1 \otimes \mathscr{W}_2]}(\mathbb{R}^{d_1+d_2})' \cong \mathcal{L}_b(\mathcal{S}^{[\mathfrak{M}_1]}_{[\mathscr{W}_1]}(\mathbb{R}^{d_1}), \mathcal{S}^{[\mathfrak{M}_2]}_{[\mathscr{W}_2]}(\mathbb{R}^{d_2})')}.$$

Theorem (Debrouwere, Neyt, and V., 2021)

Assume $\mathfrak{M}_1$, $\mathfrak{M}_2$ satisfy [L] and $[\mathfrak{M}.2]'$ and $\mathscr{W}_1$, $\mathscr{W}_2$ satisfy [wM] and [N]. Then,

$$\mathcal{S}^{[\mathfrak{M}_1\otimes\mathfrak{M}_2]}_{[\mathscr{W}_1\otimes\mathscr{W}_2]}(\mathbb{R}^{d_1+d_2}) \cong \mathcal{S}^{[\mathfrak{M}_1]}_{[\mathscr{W}_1]}(\mathbb{R}^{d_1})\widehat{\otimes}\mathcal{S}^{[\mathfrak{M}_2]}_{[\mathscr{W}_2]}(\mathbb{R}^{d_2})$$

$$\cong \mathcal{L}_b(\mathcal{S}^{[\mathfrak{M}_1]}_{[\mathscr{W}_1]}(\mathbb{R}^{d_1})', \mathcal{S}^{[\mathfrak{M}_2]}_{[\mathscr{W}_2]}(\mathbb{R}^{d_2})).$$

Consequently, we have the kernel theorem:

$$\boxed{\mathcal{S}^{[\mathfrak{M}_1\otimes\mathfrak{M}_2]}_{[\mathscr{W}_1\otimes\mathscr{W}_2]}(\mathbb{R}^{d_1+d_2})' \cong \mathcal{L}_b(\mathcal{S}^{[\mathfrak{M}_1]}_{[\mathscr{W}_1]}(\mathbb{R}^{d_1}), \mathcal{S}^{[\mathfrak{M}_2]}_{[\mathscr{W}_2]}(\mathbb{R}^{d_2})')}.$$

References

For more details, see our articles:

A. Debrouwere, L. Neyt, J. Vindas, *The nuclearity of Gelfand-Shilov spaces and kernel theorems,* Collect. Math. 72 (2021), 203–227.

A. Debrouwere, L. Neyt, J. Vindas, *Characterization of nuclearity for Beurling-Björck spaces,* Proc. Amer. Math. Soc. 12 (2020), 5171–5180.