The first factor of the class number of the p-th cyclotomic field

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Abstract. Kummer's conjecture states that the relative class number of the p-th cyclotomic field follows a strict asymptotic law. Granville has shown it unlikely to be true – it cannot be true if we assume the truth of two other widely believed conjectures. We establish a new bound for the error term in Kummer's conjecture, and more precisely we prove that $\log(h_p^-) = \frac{p+3}{4}\log p + \frac{p}{2}\log(2\pi) + \log(1-\beta) + O(\log_2 p)$, where β is a possible Siegel zero of an $L(s,\chi)$, χ odd.

Mathematics Subject Classification (2010). Primary 11R18; Secondary 11R29.

Keywords. Class numbers, Relative Class numbers, Cyclotomic Fields, Kummer's Conjecture, L-functions.

1. Introduction

Let h_p denote the class number of $\mathbb{Q}(\zeta_p)$, where p is an odd prime. Let h_p^+ denote the class number of the totally real field $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$. It is well known that h_p^+ divides h_p. We denote the quotient — or so called *first factor* of h_p — by h_p^-. The following formula is an application of the class number formula(see e.g. [9])

$$h_p^- = G(p) \prod_{\chi \bmod p, \text{odd}} L(1,\chi), \tag{1.1}$$

where $G(p) = 2p\left(\frac{p}{4\pi^2}\right)^{\frac{p-1}{4}}$. Since the magnitude of the $L(1,\chi)$ is not evident, it is hoped they are insignificant. The guess that h_p^- is asymptotically equivalent to $G(p)$ is known as Kummer's Conjecture. It is opportune to study the logarithm of this equation because the orthogonality property of characters gives us

$$\sum_{\chi \bmod p,\ \mathrm{odd}} \log(L(s,\chi)) = \frac{p-1}{2}\left(\sum_{q^m\equiv 1(p)} \frac{1}{mq^{ms}} - \sum_{q^m\equiv -1(p)} \frac{1}{mq^{ms}}\right). \quad (1.2)$$

One can estimate this sum to the right of $s = 1$, where good estimates are available, and using a zero-free region of the L-functions, one can bound the derivative in a neighbourhood of $s = 1$. Masley and Montgomery [5] obtained with these key ingredients that $|\log(h_p^-/G(p))| < 7\log p$ for $p > 200$, which is strong enough to solve the class number one problem for cyclotomic fields.

Puchta [8] improved this approach by using analogous bounds on higher derivatives, and using a *near* zero-free region, namely the open ball $B(1, \frac{1}{c\log p})$ with center 1 and radius $\frac{1}{c\log p}$, where c is some big enough constant. This is a zerofree region for all but possibly one L-function mod p, which then is necessarily quadratic and has one zero β in this region, which is necessarily real and simple and goes by the name of a Siegel zero. It is worth mentioning that if $p = 1 \bmod 4$, the odd characters are not quadratic, hence have no Siegel zero. Puchta obtained $\log(h_p^-/G(p)) = \log(1-\beta) + O((\log_2 p)^2)$.

Our proof will follow the main ideas from [8], but our practical implementation in section 3 is of a different nature, and yields

Theorem 1.1. *If no Siegel zero is present among the odd Dirichlet L-functions of conductor p, then the relative class number of $\mathbb{Q}(\zeta_p)$ satisfies*

$$|\log(h_p^-/G(p))| \le 2\log_2(p) + O(\log_3(p))$$

If there is a Siegel zero β present among the odd Dirichlet L-functions of conductor p, then the relative class number of $\mathbb{Q}(\zeta_p)$ satisfies

$$|\log(h_p^-/G(p)) - \log(1-\beta)| \le 4\log_2(p) + O(\log_3(p))$$

Since $\log(1-\beta)$ is negative, an upper bound without this term may be deduced. Finally, we note that this result sharpens the best known estimate, by Lepistö [3]. Indeed, he proves an upper bound for $\log(h_p^-/G(p))$ with main term $5\log_2(p)$.

2. Bounds around $s = 1$

In this section we exploit formula (1.2), which gives a representation in terms of splitting behaviour in $\mathbb{Q}(\zeta_p)/\mathbb{Q}$. We define

$$\Pi(x,p,a) = \sum_{q^m\le x, q^m\equiv a(p)} \frac{1}{mq^m},$$

where q^m ranges over the primepowers. A Brun-Titschmarsh style bound is given by the following lemma.

Lemma 2.1. *For $x > p$, and $p > 500$ we have that*

$$\Pi(x,p,\pm 1) \le \frac{2x}{(p-1)\log(x/p)}$$

Proof. When $x \geq p^2$, we start from the following inequality (see [5], Lemma 1)

$$\Pi(x, p, \pm 1) \leq \pi(x, p, \pm 1) + \frac{4\sqrt{x}}{p} + \log x.$$

In [7], the following strong version of the Brun-Titchmarsh inequality is proven.

$$\pi(x, p, \pm 1) \leq \frac{2x}{(p-1)(\log(x/p) + 5/6)}$$

Thus we only need to prove that

$$\frac{4\sqrt{x}}{p} + \log x < \frac{2x}{(p-1)} \left(\frac{1}{\log(x/p)} - \frac{1}{\log(x/p) + 5/6} \right).$$

By setting $x = pX$, $X \geq p$, it suffices to prove that

$$g(X) := \frac{4}{\sqrt{p}} + \frac{\log(pX)}{\sqrt{X}} < h(X) := \frac{5\sqrt{X}}{3(\log X + 5/6)^2}.$$

Now, $g(X)$ decreases for $X - e^2$ and $h(X)$ increases for $X \geq e^{19/6}$, hence it suffices to check that

$$g(p) = \frac{4}{\sqrt{p}} + \frac{2\log(p)}{\sqrt{p}} < h(p) = \frac{5\sqrt{p}}{3(\log p + 5/6)^2}$$

for $p \geq 500$. Now, $g(p)$ decreases for $p \geq 2$ and $h(p)$ increases for $p \geq e^{19/6}$, hence it suffices to check that $g(500) < h(500)$, which is clear.

When $p < x < p^2$, any two primepowers in the sum $\Pi(x, p, \pm 1)$ are necessarily coprime. Indeed, their quotient would be $1 \mod p$, so at least $p + 1$, implying that the smallest one should be less than $\frac{p^2}{p+1}$. The only option then is that $p - 1 = 2^m$ and $p^2 - 1 = 2^k$, but except for $p = 3$ this is impossible. Thus, $\Pi(x, p, \pm 1) \leq N(x, Q, p, \pm 1) + \pi(Q)$, where $N(x, Q, p, a)$ is the number of integers $n \equiv a \pmod{p}$, $n \leq x$ such that n is not divisible by any prime number less then Q. We may bound $\pi(Q)$ trivially by Q, so that the quantity to be bounded is $N(x, Q, p, \pm 1) + Q$.

In the proof of the Brun-Titchmarsh inequality

$$\pi(x, q, \pm 1) \leq \frac{2x}{(p-1)\log(x/p)}$$

using the large sieve, as in [6, p.42-44], the first step is to bound $\pi(x, q, \pm 1)$ by exactly the quantity $N(x, Q, p, \pm 1) + Q$. This shows that in this range of x, the large sieve method for the Brun-Titchmarsh inequality can be applied with the same success for primepowers as for primes. □

Let us define $f(s)$ by

$$f(s) = \Big(\sum_{\chi(-1)=-1} \log L(s, \chi) \Big) - \log(s - \beta),$$

in case that any of the L-functions with χ odd has a so-called Siegel zero β in $]1 - \frac{1}{c \log p}, 1]$, where c is some big enough constant. Otherwise, we leave out the term with the Siegel zero. In any case f is holomorphic in $B(1, \frac{1}{c \log p})$.

Lemma 2.2. *For any c, $p \geq 500$, and $\sigma \in]1, 1+\frac{1}{c\log p}]$, we have the following estimates.*

$$|f(\sigma)| \leq (1+1_\beta)\log\big(\frac{1}{\sigma-1}\big)+\frac{3}{2} \tag{2.1}$$

$$|f^{(\nu)}(\sigma)| \leq \big(1+1_\beta+c_{p,\nu}\big)\frac{(\nu-1)!}{(\sigma-1)^\nu} \tag{2.2}$$

Where the notation 1_β stands for 1 if a Siegel zero is present and 0 otherwise, and we may choose the $c_{p,\nu}$ to be equal to $\frac{\log(2)}{2c^\nu(\nu-1)!\log p}+\frac{\log_2(p)+\log(c)-\log_2(2)+e^{-1}}{c^\nu(\nu-1)!}+\frac{1}{c\log p}+\frac{\sigma\lfloor\log\nu\rfloor}{\nu-\lfloor\log\nu\rfloor}+\frac{\sigma\nu}{c^{\lfloor\log\nu\rfloor}\lfloor\log\nu\rfloor!}$.

Proof. The case $\nu = 0$ can be proven as in [5]. The estimates for the derivatives are stated in [8], but the statement is slightly incorrect and the proof omitted, so we will prove them here in full. We bound the sums occurring in the ν-th derivative of (1.2) using Lemma 2.1 and partial summation.

$$\begin{aligned}
\frac{p-1}{2}\sum_{q^m\equiv 1(p)}\frac{(m\log q)^\nu}{mq^{m\sigma}} &= \frac{p-1}{2}\int_{2p}^{\infty}\frac{(\log x)^\nu d(\Pi(x,p,1))}{x^\sigma}\\
&= \frac{p-1}{2}\int_{2p}^{\infty}\frac{\sigma x^{\sigma-1}(\log x)^\nu-\nu x^{\sigma-1}(\log x)^{\nu-1}}{x^{2\sigma}}\Pi(x,p,1)dx\\
&\leq \int_{2p}^{\infty}\frac{\sigma(\log x)^\nu}{x^\sigma\log(x/p)}dx\\
&= \frac{p\sigma}{p^\sigma}\int_{2}^{\infty}\frac{(\log x+\log p)^\nu}{x^\sigma\log x}dx =: I,
\end{aligned}$$

where we possibly omitted the first term $\frac{(p-1)\log(p+1)^\nu}{2m(p+1)^\sigma}$ if $p+1$ is a primepower q^m. If this is the case, then $q = 2$ and $m = \log(p+1)/\log(2)$. This term is smaller than $\varepsilon_1\frac{(\nu-1)!}{(\sigma-1)^\nu}$ for all σ in the desired range for $\varepsilon_1 = \frac{\log(2)}{2c^\nu(\nu-1)!\log p}$. We expand the integrand with the binomial theorem, and get

$$\begin{aligned}
I &= \frac{p\sigma}{p^\sigma}(\log p)^\nu\int_2^\infty\frac{1}{x^\sigma\log x}dx+\frac{p\sigma}{p^\sigma}\sum_{i=0}^{\nu-1}\frac{\nu!(\log p)^i}{(\nu-i)!i!}\int_1^\infty\frac{(\log x)^{\nu-i-1}}{x^\sigma}dx\\
&= \frac{p\sigma}{p^\sigma}(\log p)^\nu\int_2^\infty\frac{1}{x^\sigma\log x}dx+\frac{(\nu-1)!}{(\sigma-1)^\nu}\frac{p\sigma}{p^\sigma}\sum_{i=0}^{\nu-1}\frac{\nu}{\nu-i}\frac{((\sigma-1)\log p)^i}{i!},
\end{aligned}$$

where we have used the identity

$$\int_1^\infty\frac{(\log x)^a}{x^\sigma}dx=\int_0^\infty\frac{t^a}{e^{(\sigma-1)t}}dt=\frac{a!}{(\sigma-1)^{a+1}}.$$

We consider first the term

$$\frac{p\sigma}{p^\sigma}(\log p)^\nu \int_2^\infty \frac{1}{x^\sigma \log x} dx = \frac{p\sigma}{p^\sigma}(\log p)^\nu \int_{\log 2}^\infty e^{-(\sigma-1)t}\frac{dt}{t}$$
$$\leq \frac{p\sigma}{p^\sigma}(\log p)^\nu \left(\int_{(\sigma-1)\log 2}^1 \frac{1}{t} dt + \int_1^\infty e^{-t} dt\right)$$
$$\leq (\log p)^\nu \left(\log(\frac{1}{\sigma-1}) - \log_2(2) + e^{-1}\right).$$

Because $p\sigma \leq p^\sigma$. We now seek the ε_2 such that

$$(\log p)^\nu \left(\log(\frac{1}{\sigma-1}) - \log_2(2) + e^{-1}\right) \leq \varepsilon_2 \frac{(\nu-1)!}{(\sigma-1)^\nu}.$$

If we put $\varepsilon_2 = \frac{\log_2(p)+\log(c)-\log_2(2)+e^{-1}}{c^\nu(\nu-1)!}$, the inequality holds for for $\sigma \to 1$ and for $\sigma = 1 + \frac{1}{c\log p}$. One may check that the derivative of the difference does not have a zero in the interval under consideration if $p > e^e$. Thus the difference is monotone, and the inequality holds throughout.

To deal with the rest of the terms efficiently, write $X = (\sigma-1)\log p \leq 1/c$. Then we have for any integer $B \geq 1$

$$\frac{p\sigma}{p^\sigma}\sum_{i=0}^{\nu-1}\frac{\nu}{\nu-i}\frac{X^i}{i!} \leq \frac{p\sigma}{p^\sigma}\sum_{i=0}^{B-1}\frac{\nu}{\nu-B}\frac{X^i}{i!} + \frac{p\sigma}{p^\sigma}X^B\sum_{i=0}^{\nu-1}\frac{\nu}{B!}\frac{X^{i-B}}{(i-B)!}$$
$$\leq \frac{p\sigma}{p^\sigma}\frac{\nu}{\nu-B}e^X + \frac{p\sigma}{p^\sigma}\frac{\nu}{c^\nu B!}e^X = \frac{\nu\sigma}{\nu-B} + \frac{\nu\sigma}{c^B B!}$$

We now put $B = \lfloor\log\nu\rfloor$, and see that the to be bounded sum is bounded by $(1+\varepsilon_3)\frac{(\nu-1)!}{(\sigma-1)^\nu}$, where $\varepsilon_3 = \frac{1}{c\log p} + \frac{\sigma\lfloor\log\nu\rfloor}{\nu-\lfloor\log\nu\rfloor} + \frac{\sigma\nu}{c^{\lfloor\log\nu\rfloor}\lfloor\log\nu\rfloor!}$

One may now bound the $\varepsilon_1+\varepsilon_2+\varepsilon_3$ by the coefficient of $\frac{(\nu-1)!}{(\sigma-1)^\nu}$ except the 1_β in the statement of the lemma. We note that the sum over the primepowers congruent to -1 mod p obeys the same bound with the same proof as above. One of the sums is strictly positive and the other is strictly negative, thus we have proven that

$$|f^\nu(s) + (\log(\sigma-\beta))^{(\nu)}| \leq (1+c_{p,\nu})\frac{(\nu-1)!}{(\sigma-1)^\nu},$$

or since $\frac{(\nu-1)!}{(\sigma-\beta)^\nu} \leq \frac{(\nu-1)!}{(\sigma-1)^\nu}$,

$$|f^\nu(s)| \leq (1+1_\beta+c_{p,\nu})\frac{(\nu-1)!}{(\sigma-1)^\nu}.$$

□

On the other hand we can prove the following bound on the derivatives of f to the right of $s=1$, using the holomorphic property of f on $B(1,\frac{1}{c\log p})$, when c is big enough. We note that due to Kadiri ([2], Theorem 12.1) the value $c = 6.4355$ is big enough.

Lemma 2.3. *For* $c > 6.4355$, $\frac{p-1}{\log p} > c$, *and* $\sigma \in [1, 1 + \frac{2}{c \log p}]$, *we have that*

$$|f^{(\nu)}(\sigma)| \leq 2c^\nu \nu!\, p \log^{\nu+1} p \tag{2.3}$$

Proof. Recall the lemma of Borel-Caratheodory (see [1], p. 12) which states that if g is holomorphic and $\Re(g(s)) \leq M$ in $B(\sigma_0, R)$ and $g(\sigma_0) = 0$, then

$$|g^\nu(s)| \leq \frac{2M\nu!}{(R-r)^\nu}, \qquad s \in B(\sigma_0, r).$$

We wish to apply this to $f(s) - f(\sigma_0)$. This function vanishes at σ_0, and is holomorphic as long as $R \leq \sigma_0 - (1 - \frac{1}{c \log p})$. For the bound on the real part, consider

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = s \int_1^\infty \frac{\sum_{n \leq x} \chi(n)}{x^{s+1}} dx.$$

Since $|\sum_{n=1}^{x} \chi(n)| \leq \frac{p}{2}$, we have that $|L(s,\chi)| \leq |s| \int_1^\infty \frac{|\sum_{n\leq x} \chi(n)|}{x^{\sigma+1}} dx \leq \frac{|s|p}{2\sigma}$. This means that

$$\Re(f(s)) \leq \frac{p-1}{2}(\log p + \log(|s|/2\sigma)) - \log(|s - \beta|).$$

For s on the border of the domain determined by $3/4 < \Re(s) < 2$, $|\Im(s)| \leq \frac{1}{4}$, $|s|/2\sigma \leq \sqrt{10}/6$ and say $|s - \beta| > 1/8$, thus this bound is smaller than $\frac{p-1}{2} \log p$. Since $f(s)$ is harmonic with at most logarithmic singularities in which $\Re(f) \to -\infty$, the same bound holds also inside the domain. In the region $\sigma > 1$, consider the following estimation.

$$|\Re(\log L(s,\chi))| = |\Re\Big(\sum_{q^m} \frac{\chi(q^m)}{mq^{ms}}\Big)| \leq \sum_{q^m} \frac{1}{mq^{ms}} = \log \zeta(\sigma) \leq \log(\frac{\sigma}{\sigma - 1}),$$

thus if $\sigma_0 > p/(p-1)$, then $|\Re(f(\sigma_0))| \leq \frac{p-1}{2} \log(p) + \log(p-1)$. In conclusion, as long as $\sigma_0 > p/(p-1)$,

$$\Re(f(\sigma) - f(\sigma_0)) \leq p \log p.$$

One retrieves the statement of the theorem by putting $\sigma_0 = 1 + \frac{1}{c \log p}$, $R = \frac{2}{c \log p}$, $r = \frac{1}{c \log p}$. □

3. Worst case scenario

Among all functions that satisfy the bounds from the preceding section, what is the largest value $f(1)$ can attain? We define σ_ν to be the point where the bound (2.2) and the absolute bound (2.3) coincide. We note that

$$\sigma_\nu - 1 = \frac{1}{c \log p} \sqrt[\nu]{\frac{1 + 1_\beta + c_{p,\nu}}{2\nu p \log p}} \geq \frac{1}{c \log p \sqrt[\nu]{2\nu p \log p}}. \tag{3.1}$$

Theorem 3.1. *For all $p > 500$, and $c > 6.4355$,*

$$|f(1)| \leq (1 + 1_\beta.2 + e^{1/c}) \log_2(p) + O(1),$$

where the $O(1)$-term is bounded by $(3+e^{1/c})\log(c)+0.791e^{1/c}+10.720+\frac{0.943}{c}$

Proof. We use the Taylor expansion of f with error term in integral form,

$$f(1) = f(\sigma_\nu) + (1-\sigma_\nu)f'(\sigma_\nu) + \frac{(1-\sigma_\nu)^2}{2} f^{(2)}(\sigma_\nu) + ...$$
$$+ \int_{\sigma_\nu}^{1} \frac{f^{(\nu)}(x)}{(\nu-1)!}(1-x)^{\nu-1}dx$$

Now note that $|f^{(\nu)}(x)|$ is bounded above by the bound (2.3) for all x between 1 and σ_ν, which is equal to $|f^{(\nu)}(\sigma_\nu)|$. Using (2.1), (2.2) and (3.1), we get

$$|f(1)| \leq |f(\sigma_\nu)| + \sum_{i=1}^{\nu} \frac{(\sigma_\nu - 1)^i}{i!} |f^{(i)}(\sigma_\nu)|.$$
$$\leq (1+1_\beta)\log(\frac{1}{\sigma_\nu - 1}) + 3/2 + \sum_{i=1}^{\nu} \frac{1 + 1_\beta + c_{p,i}}{i}$$
$$\leq (1+1_\beta)\Big(\log_2(p) + \log(c) + \frac{\log(2\nu p \log p)}{\nu}\Big) + 3/2 + \sum_{i=1}^{\nu} \frac{1 + 1_\beta + c_{p,i}}{i}.$$

Upon taking $\nu = \log p$, this first contribution is bounded by

$$(1+1_\beta)\Big(\log_2(p) + \log(c) + 1 + \frac{\log(2(\log p)^2)}{\log p}\Big) + 3/2.$$

In the rest of the terms, we find the first ν terms of some converging series;

$$\sum_{i=1}^{\nu} \frac{1}{c^i i!} \leq e^{1/c} - 1, \quad \sum_{i=1}^{\nu} \frac{\lfloor \log \nu \rfloor}{\nu(\nu - \lfloor \log \nu \rfloor)} \leq 1.90, \quad \sum_{i=1}^{\nu} \frac{1}{c^{\lfloor \log \nu \rfloor} \lfloor \log \nu \rfloor !} \leq 1.13.$$

Using this and the well-known estimate $\sum_{i=1}^{\nu} \frac{1}{i} \leq \log(\nu) + 1$ we bound the last contribution as follows

$$\sum_{i=1}^{\nu} \frac{1 + 1_\beta + c_{p,i}}{i} \leq (1 + 1_\beta + \frac{1}{c \log p})(\log(\nu) + 1) + (1 + \frac{1}{c\log p})3.03$$
$$+ \Big(\frac{\log(2)}{2\log p} + \log_2(p) + \log(c) - \log_2(2) + e^{-1}\Big)(e^{1/c} - 1).$$

Gathering everything and filling in $p = 500$ for the terms converging to zero, we recover the statement of the theorem. □

To finish the proof of Theorem 1.1, one now needs to plug the above estimate of $f(1)$ into the logarithm of the formula (1.1), and check that the choice of $c = \log_2(p)\frac{6.4355}{\log_2(500)}$ is permitted.

Remark 3.2. It is quite counterintuitive that a bigger value of c gives a better estimate in Theorem 3.1 while a smaller value of c means a bigger zero-free region, thus means a stronger input. In truth there is a tradeoff between having σ_ν *big* to control the main term coming from Lemma 2.2 and at the same time *not too big* to bound the term coming from ε_1 in the proof of Lemma 2.2. This ε_1 cannot be efficiently bounded by a lack of good bounds on the number of primes of the form $ap+1$, where a is a small integer.

Remark 3.3. From (1.1) it is now clear that the general behaviour of h_p^- is dominated by $G(p)$ and that the L-values can perturb this term only slightly. It is somewhat common(see [4]) to state upper bounds for h_p^- in terms of $G(p)$, where $4\pi^2$ is replaced by a smaller constant.

Corollary 3.4. *We have that* $h_p^- \leq 2p\left(\frac{p}{39}\right)^{\frac{p-1}{4}}$, *for* $p > 9649$.

Proof. This follows from plugging in $c = \frac{6.4355\log_2(p)}{\log_2(500)} = 3.523\log_2(p)$ in Theorem 3.1 and checking that

$$|f(1)| \leq e^{\frac{p-1}{4}} \log(\frac{4\pi^2}{39}),$$

whenever $p > 9649$. □

References

[1] T. Estermann. *Introduction to Modern Prime Number Theory*, volume 41 of *Cambridge Tracts in Mathematics and Mathematical Physics*. Cambridge University Press, 1961.

[2] H. Kadiri. Régions explicites sans zéros pour les fonctions l de dirichlet, phd thesis.

[3] Timo Lepistö. On the growth of the first factor of the class number of the prime cyclotomic field. *Ann. Acad. Sci. Fenn. Ser. A I*, (577):21, 1974.

[4] Stéphane R. Louboutin. Mean values of L-functions and relative class numbers of cyclotomic fields. *Publ. Math. Debrecen*, 78(3-4):647–658, 2011.

[5] J. Myron Masley and Hugh L. Montgomery. Cyclotomic fields with unique factorization. *J. Reine Angew. Math.*, 286/287:248–256, 1976.

[6] H. L. Montgomery. *Topics in Multiplicative Number Theory*, volume 227 of *Lecture Notes in Mathematics*. Springer-Verlag, 1971.

[7] H. L. Montgomery and R. C. Vaughan. The large sieve. *Mathematika*, 20:119–134, 1973.

[8] Jan-Christoph Puchta. On the class number of p-th cyclotomic field. *Arch. Math. (Basel)*, 74(4):266–268, 2000.

[9] L. C. Washington. *Introduction to Cyclotomic Fields*, volume 83 of *Graduate Texts in Mathematics*. Springer, 1982.

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