Nonlocal problems for superconductivity

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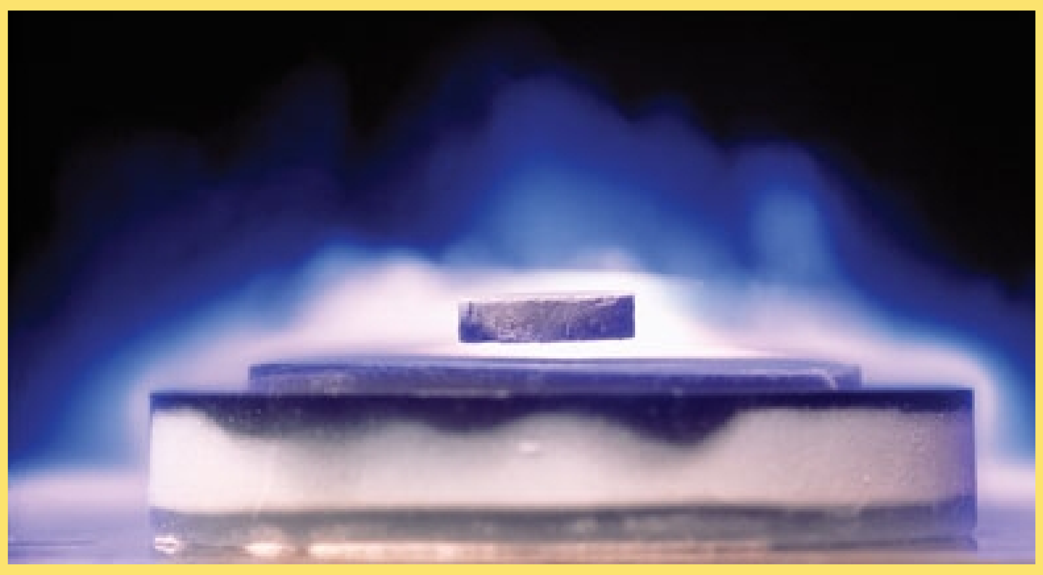
Domain

- ▷ Bounded domain Ω in $\mathbb{R}^3$ with a Lipschitz continuous boundary Γ
- ▷ $\boldsymbol{\nu}$ denotes the outward unit normal vector on Γ
- ▷ Ω is occupied by a superconductive material
- ▷ This is a material, which loses all resistivity below a certain temperature T_c
- ▷ Figure: a magnet levitates above a ceramic superconductor cooled by liquid nitrogen



Full Maxwell's equations ($\delta = 1$) and quasi-static Maxwell's equations ($\delta = 0$) for linear materials

$$\nabla \times \mathbf{H} = \mathbf{J} + \delta\varepsilon\partial_t\mathbf{E}$$
$$\nabla \times \mathbf{E} = -\mu\partial_t\mathbf{H}$$
$$\nabla \cdot \mathbf{H} = 0$$

$\mathbf{H}$	magnetic field
$\mathbf{E}$	electric field
$\mathbf{J}$	current density
$\varepsilon > 0$	electric permittivity
$\mu > 0$	magnetic permeability

Two-fluid model

$$\mathbf{J} = \mathbf{J}_n + \mathbf{J}_s$$
$$\mathbf{J}_n = \sigma\mathbf{E} \qquad \text{Ohm's law}$$

$$\nabla \times \mathbf{H} = \sigma\mathbf{E} + \mathbf{J}_s + \delta\varepsilon\partial_t\mathbf{E}$$
$$\nabla \times \mathbf{E} = -\mu\partial_t\mathbf{H}$$
$$\nabla \cdot \mathbf{H} = 0$$

$\mathbf{J}_n$	normal current density
$\mathbf{J}_s$	superconducting current density
σ	conductivity of normal electrons

Local law for $\mathbf{J}_s$: London equations (1935)

$$\partial_t\mathbf{J}_s = \Lambda^{-1}\mathbf{E}$$
$$\nabla \times \mathbf{J}_s = -\Lambda^{-1}\mathbf{B}$$
$$\Lambda = \frac{m}{n_s q^2}$$

n_s	density of superelectrons
m	mass of superelectrons
q	electric charge of superelectrons

$\Rightarrow$ Correct description of two basic properties of superconductors: perfect conductivity and perfect diamagnetism (Meissner effect)

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \exists \mathbf{A} \in \mathbf{H}^1(\Omega) \text{ such that } \mathbf{B} = \nabla \times \mathbf{A} \text{ and } \nabla \cdot \mathbf{A} = 0$$

$$\nabla \times \mathbf{J}_s = -\Lambda^{-1}\mathbf{B}$$
$$\Downarrow$$
$$\mathbf{J}_s(\mathbf{x}, t) = -\Lambda^{-1}\mathbf{A}(\mathbf{x}, t), \qquad (\mathbf{x}, t) \in \Omega \times (0, T)$$

Generalization: nonlocal laws

▷ Pippard (1953)

$$\mathbf{J}_{s,p}(\mathbf{x}, t) = \int_\Omega Q(\mathbf{x} - \mathbf{x}')\mathbf{A}(\mathbf{x}', t)\,\mathrm{d}\mathbf{x}', \qquad (\mathbf{x}, t) \in \Omega \times (0, T)$$

with

$$Q(\mathbf{x} - \mathbf{x}')\mathbf{A}(\mathbf{x}', t) = -C\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^4}\left[\mathbf{A}(\mathbf{x}', t) \cdot (\mathbf{x} - \mathbf{x}')\right] \exp\left(-\frac{|\mathbf{x} - \mathbf{x}'|}{r_0}\right)$$

▷ $C > 0$, $r_0 > 0$ is related to the mean free path in the material

▷ Eringen (1984)

$$\mathbf{J}_{s,e}(\mathbf{x}, t) = \int_\Omega \sigma_0\left(|\mathbf{x} - \mathbf{x}'|\right)(\mathbf{x} - \mathbf{x}') \times \mathbf{H}(\mathbf{x}', t)\,\mathrm{d}\mathbf{x}', \qquad (\mathbf{x}, t) \in \Omega \times (0, T)$$

with

$$\sigma_0(s) = \begin{cases} \frac{C}{2s^2}\exp\left(-\frac{s}{r_0}\right) & s < r_0; \\ 0 & s \geqslant r_0 \end{cases}$$

Lemma

$$(\mathbf{x}, t) \in \Omega \times (0, T), \mathbf{H} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma \Rightarrow \nabla \times \mathbf{J}_{s,e}(\mathbf{x}, t) = -\int_\Omega K(\mathbf{x} - \mathbf{x}')\mathbf{H}(\mathbf{x}', t)\,\mathrm{d}\mathbf{x}' = -(K \star \mathbf{H})(\mathbf{x}, t) \text{ with } K : \Omega \to \mathbb{R} : \mathbf{y} \mapsto \begin{cases} \frac{C}{2|\mathbf{y}|^2}\left(1 - \frac{|\mathbf{y}|}{r_0}\right)\exp\left(-\frac{|\mathbf{y}|}{r_0}\right) & |\mathbf{y}| < r_0; \\ 0 & |\mathbf{y}| \geqslant r_0 \end{cases}$$

Consequence: two models ($\delta = 0, 1$)

$$\nabla \times \nabla \times \mathbf{H} = \sigma\nabla \times \mathbf{E} + \nabla \times \mathbf{J}_{s,e} + \delta\varepsilon\nabla \times \partial_t\mathbf{E} \quad \Rightarrow \quad \delta\varepsilon\mu\partial_{tt}\mathbf{H} + \sigma\mu\partial_t\mathbf{H} - \Delta\mathbf{H} + K \star \mathbf{H} = \mathbf{0}$$

Solution method

- ▷ Time discretization is based on Backward Euler's method
- ▷ Rothe's method $\Rightarrow$ unique weak solution for both problems

Numerical Experiment

- ▷ Penetration of magnetic field and induced current into the material (unit cube)
- ▷ Main difficulty: programming convolution term in Fenics (not finished)

Further research

- ▷ Comparison of the two models
- ▷ Nonlinear model for $\mathbf{E}$ based on the power law $\mathbf{J}_n(\mathbf{E}) = |\mathbf{E}|^{-\alpha}\mathbf{E}, \alpha \in (0, 1)$

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