A nonlocal parabolic model for type-I superconductors

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Domain

- ▷ Bounded domain Ω in $\mathbb{R}^3$ with a Lipschitz continuous boundary Γ
- ▷ $\boldsymbol{\nu}$ denotes the outward unit normal vector on Γ
- ▷ Ω is occupied by a **superconductive material** of type-I
- ▷ This is a material, which loses all resistivity below a certain temperature T_c

Quasi-static Maxwell's equations for linear materials

$\nabla \times \mathbf{H} = \mathbf{J}$	Ampère's law
$\nabla \times \mathbf{E} = -\mu \partial_t \mathbf{H}$	Faraday's law
$\nabla \cdot \mathbf{H}_0 = 0$	

$\mathbf{H}$	magnetic field
$\mathbf{E}$	electric field
$\mathbf{J}$	current density
$\varepsilon > 0$	electric permittivity
$\mu > 0$	magnetic permeability

Two-fluid model

$\mathbf{J} = \mathbf{J}_n + \mathbf{J}_s$	
$\mathbf{J}_n = \sigma \mathbf{E}$	Ohm's law

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \mathbf{J}_s$$
$$\nabla \times \mathbf{E} = -\mu \partial_t \mathbf{H}$$
$$\nabla \cdot \mathbf{H}_0 = 0$$

$\mathbf{J}_n$	normal current density
$\mathbf{J}_s$	superconducting current density
σ	conductivity of normal electrons

Local law for $\mathbf{J}_s$: London equations (1935)

$$\partial_t \mathbf{J}_s = \Lambda^{-1}\mathbf{E}$$
$$\nabla \times \mathbf{J}_s = -\Lambda^{-1}\mathbf{B}$$
$$\Lambda = \frac{m_e}{n_s e^2}$$

n_s	density of superelectrons
m_e	mass of an electron
q	electric charge of an electron

$\Rightarrow$ Correct description of two basic properties of superconductors:
perfect conductivity and perfect diamagnetism (Meissner effect)

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \exists \mathbf{A} \in \mathbf{H}^1(\Omega) \text{ such that } \mathbf{B} = \nabla \times \mathbf{A} \text{ and } \nabla \cdot \mathbf{A} = 0$$

$$\nabla \times \mathbf{J}_s = -\Lambda^{-1}\mathbf{B}$$
$$\Downarrow$$
$$\mathbf{J}_s(\mathbf{x},t) = -\Lambda^{-1}\mathbf{A}(\mathbf{x},t), \qquad (\mathbf{x},t) \in \Omega \times (0,T)$$

Generalization: nonlocal laws

▷ Pippard (1953)

$$\mathbf{J}_{s,p}(\mathbf{x},t) = \int_\Omega Q(\mathbf{x}-\mathbf{x}')\mathbf{A}(\mathbf{x}',t)\,\mathrm{d}\mathbf{x}', \qquad (\mathbf{x},t) \in \Omega \times (0,T)$$

with

$$Q(\mathbf{x}-\mathbf{x}')\mathbf{A}(\mathbf{x}',t) = -\widetilde{C}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^4}[\mathbf{A}(\mathbf{x}',t)\cdot(\mathbf{x}-\mathbf{x}')]\exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|}{r_0}\right)$$

▷ $\widetilde{C} > 0$, $r_0 > 0$ is related to the mean free path in the material

▷ Eringen (1984)

$$\mathbf{J}_{s,e}(\mathbf{x},t) = \int_\Omega \sigma_0(|\mathbf{x}-\mathbf{x}'|)(\mathbf{x}-\mathbf{x}') \times \mathbf{H}(\mathbf{x}',t)\,\mathrm{d}\mathbf{x}' =: -(\mathcal{K}_0 \star \mathbf{H})(\mathbf{x},t), \qquad (\mathbf{x},t) \in \Omega \times (0,T)$$

with

$$\sigma_0(s) = \begin{cases} \frac{\widetilde{C}}{2s^2}\exp\left(-\frac{s}{r_0}\right) & s < r_0; \\ 0 & s \geqslant r_0 \end{cases}$$

Consequence: model 1

$$\nabla \times \nabla \times \mathbf{H} = \sigma \nabla \times \mathbf{E} + \nabla \times \mathbf{J}_{s,e} \qquad \Rightarrow \qquad \sigma\mu\partial_t \mathbf{H} + \nabla \times \nabla \times \mathbf{H} + \nabla \times (\mathcal{K}_0 \star \mathbf{H}) = \mathbf{0}$$

Variational formulation: $(\partial_t \mathbf{H}, \boldsymbol{\varphi}) + (\nabla \times \mathbf{H}, \nabla \times \boldsymbol{\varphi}) + (\mathcal{K}_0 \star \mathbf{H}, \nabla \times \boldsymbol{\varphi}) = (\mathbf{f}, \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ (1)

Solution method: Rothe's method

▷ Divide the interval $[0,T]$ into n equidistant subintervals $[t_{i-1}, t_i]$ with time step $\tau = \frac{T}{n}$

▷ Notation

$$\mathbf{h}_i = \mathbf{H}(t_i), \qquad \delta\mathbf{h}_i = \frac{\mathbf{h}_i - \mathbf{h}_{i-1}}{\tau}, \qquad \mathbf{h}_n(t) = \begin{cases} \mathbf{H}_0 & \text{if } t = 0; \\ \mathbf{h}_{i-1} + (t - t_{i-1})\delta\mathbf{h}_i & \text{for } t \in (t_{i-1}, t_i], \quad i = 1, \ldots, n \end{cases}$$

Numerical scheme model 1a: convolution implicit

$$\begin{cases} (\delta\mathbf{h}_i, \boldsymbol{\varphi}) + (\nabla \times \mathbf{h}_i, \nabla \times \boldsymbol{\varphi}) + (\mathcal{K}_0 \star \mathbf{h}_i, \nabla \times \boldsymbol{\varphi}) = (\mathbf{f}_i, \boldsymbol{\varphi}), & \boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}, \Omega); \\ \mathbf{h}_0 = \mathbf{H}_0 \end{cases}$$

Numerical scheme model 1b: convolution explicit

$$\begin{cases} (\delta\mathbf{h}_i, \boldsymbol{\varphi}) + (\nabla \times \mathbf{h}_i, \nabla \times \boldsymbol{\varphi}) = (\mathbf{f}_i, \boldsymbol{\varphi}) - (\mathcal{K}_0 \star \mathbf{h}_{i-1}, \nabla \times \boldsymbol{\varphi}), & \boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}, \Omega); \\ \mathbf{h}_0 = \mathbf{H}_0 \end{cases}$$

Theorem: Existence existence and uniqueness solution model 1a

Let $\mathbf{H}_0 \in \mathbf{L}^2(\Omega)$ and $\mathbf{f} \in L^2\left((0,T), \mathbf{L}^2(\Omega)\right)$. Assume that $\nabla \cdot \mathbf{H}_0 = 0 = \nabla \cdot \mathbf{f}(t)$ for any time $t \in [0,T]$. Then there exists a solution $\mathbf{H} \in C([0,T], \mathbf{L}^2(\Omega)) \cap L^2((0,T), \mathbf{H}^{\frac{1}{2}}(\Omega))$ with $\partial_t \mathbf{H} \in L_2\left((0,T), \mathbf{H}_0^{-1}(\mathbf{curl}, \Omega)\right)$, which solves (1)

Lemma: new convolution kernel

$$(\mathbf{x},t) \in \Omega \times (0,T), \nabla \cdot \mathbf{H} = 0 \text{ and } \mathbf{H} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma \quad \Rightarrow \quad \nabla \times \mathbf{J}_{s,e}(\mathbf{x},t) = -\int_\Omega \mathcal{K}(\mathbf{x}-\mathbf{x}')\mathbf{H}(\mathbf{x}',t)\,\mathrm{d}\mathbf{x}' =: -(\mathcal{K} \star \mathbf{H})(\mathbf{x},t),$$

where $\mathcal{K} : \Omega \times \Omega \to \mathbb{R} : (\mathbf{x}, \mathbf{x}') \mapsto \kappa(|\mathbf{x}-\mathbf{x}'|)$ with $\kappa : (0,\infty) \to \mathbb{R} : s \mapsto \begin{cases} \frac{\widetilde{C}}{2s^2}\left(1 - \frac{s}{r_0}\right)\exp\left(-\frac{s}{r_0}\right) & s < r_0; \\ 0 & s \geqslant r_0 \end{cases}$

Consequence: model 2

$$\nabla \times \nabla \times \mathbf{H} = \sigma \nabla \times \mathbf{E} + \nabla \times \mathbf{J}_{s,e} \qquad \Rightarrow \qquad \sigma\mu\partial_t \mathbf{H} - \Delta \mathbf{H} + \mathcal{K} \star \mathbf{H} = \mathbf{0}$$

Variational formulation: $(\partial_t \mathbf{H}, \boldsymbol{\varphi}) + (\nabla \mathbf{H}, \nabla \boldsymbol{\varphi}) + (\mathcal{K} \star \mathbf{H}, \boldsymbol{\varphi}) = (\mathbf{f}, \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega)$

Numerical scheme model 2a: convolution implicit

$$\begin{cases} (\delta\mathbf{h}_i, \boldsymbol{\varphi}) + (\nabla \mathbf{h}_i, \nabla \boldsymbol{\varphi}) + (\mathcal{K} \star \mathbf{h}_i, \boldsymbol{\varphi}) = (\mathbf{f}_i, \boldsymbol{\varphi}), & \boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega); \\ \mathbf{h}_0 = \mathbf{H}_0 \end{cases}$$

Numerical scheme model 2b: convolution explicit

$$\begin{cases} (\delta\mathbf{h}_i, \boldsymbol{\varphi}) + (\nabla \mathbf{h}_i, \nabla \boldsymbol{\varphi}) = (\mathbf{f}_i, \boldsymbol{\varphi}) - (\mathcal{K} \star \mathbf{h}_{i-1}, \boldsymbol{\varphi}), & \boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega); \\ \mathbf{h}_0 = \mathbf{H}_0 \end{cases}$$

Theorem: Error estimates model 2a

▷ If $\mathbf{H}_0 \in \mathbf{H}_0^1(\Omega)$ then

$$\max_{t \in [0,T]} \|\mathbf{h}_n(t) - \mathbf{H}(t)\|^2 + \int_0^T \|\nabla[\mathbf{h}_n - \mathbf{H}]\|^2 \leqslant C\tau$$

▷ If $\mathbf{H}_0 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$ then

$$\max_{t \in [0,T]} \|\mathbf{h}_n(t) - \mathbf{H}(t)\|^2 + \int_0^T \|\nabla[\mathbf{h}_n - \mathbf{H}]\|^2 \leqslant C\tau^2$$

Further research

▷ Performing numerical experiments using the finite element library DOLFIN from the FEniCS project

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