Determination of an unknown diffusion coefficient in a parabolic problem

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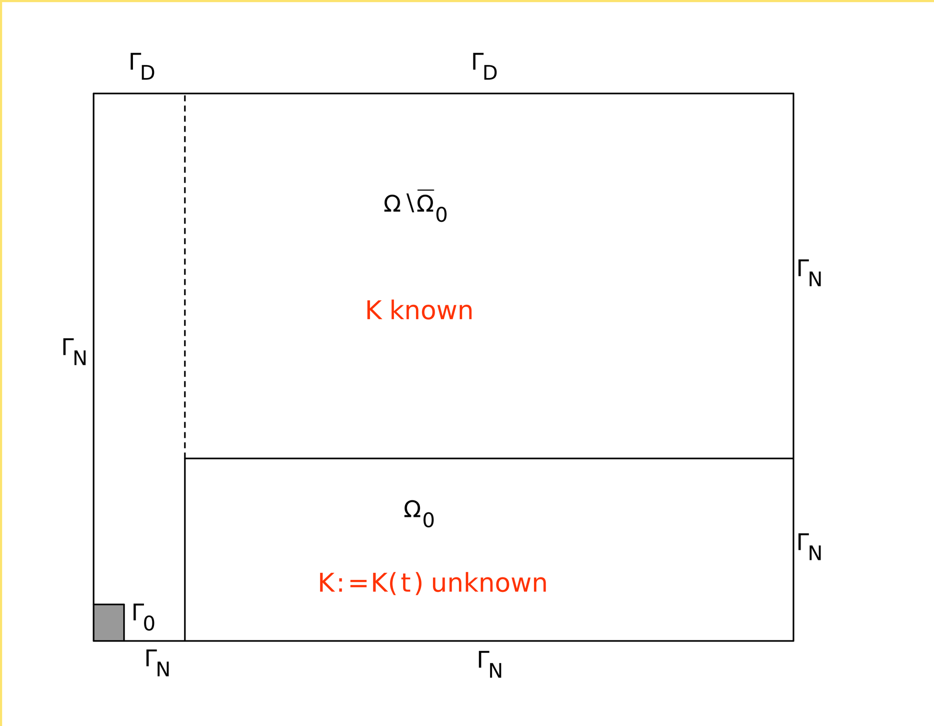
Application

- ▷ Spontaneous potential well-logging is an important technique to detect parameters (e.g. resistivity, diffusivity) of the formation in petroleum exploitation;
- ▷ The resistivity can depend on temperature and humidity in some geological formations. This makes the problem of the resistivity identification time-dependent.



Mathematical model (IBVP)

Find (K, u) such that ($T > 0$ fixed)

$$\begin{aligned} \partial_t u - \nabla \cdot (K \nabla u) &= f(u) && \text{in } (0,T) \times \Omega; \\ u &= g^D && \text{in } (0,T) \times \Gamma_D; \\ -K \nabla u \cdot \boldsymbol{\nu} &= g^N && \text{in } (0,T) \times \Gamma_N; \\ u(0) &= u_0 && \text{in } \Omega; \end{aligned}$$

with the following conditions on Γ_0

$$\begin{cases} \int_{\Gamma_0} -K \nabla u \cdot \boldsymbol{\nu} = h(t) & \text{in } (0,T); \\ u = U(t) & \text{on } (0,T) \times \Gamma_0. \end{cases}$$

Solution method: time discretization

The time discretization is based on Backward Euler's method. Divide the time interval $[0, T]$ into $n \in \mathbb{N}$ equidistant subintervals (t_{i-1}, t_i) for $t_i = i\tau$, where $\tau = T/n$ and introduce the following notation for any function z

$$z_i = z(t_i), \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}.$$

First solution method: determine a solution of

$$\begin{aligned} \delta u_i - \nabla \cdot (K_i \nabla u_i) &= f(u_{i-1}) && \text{in } \Omega; \\ u_i &= 0 && \text{on } \Gamma_D; \\ -K_i \nabla u_i \cdot \boldsymbol{\nu} &= 0 && \text{on } \Gamma_N; \\ \int_{\Gamma_0} -K_i \nabla u_i \cdot \boldsymbol{\nu} &= h_i \end{aligned}$$

for a given K_i, $i = 1, \ldots, n$. Search K_i such that $u_i|_{\Gamma_0} = U_i$.

Second solution method: solve

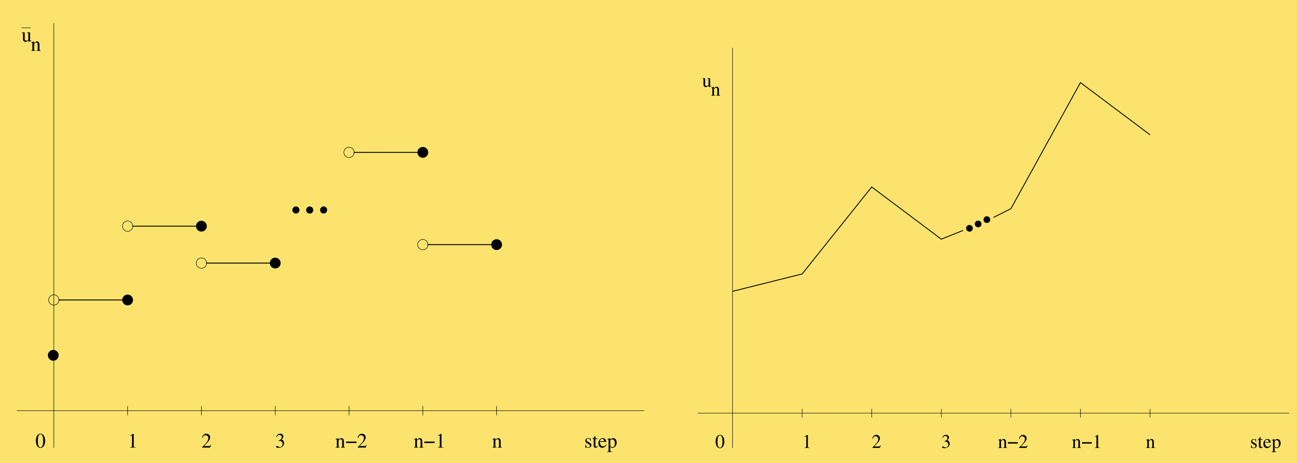
$$\begin{aligned} \delta u_i - \nabla \cdot (K_i \nabla u_i) &= f(u_{i-1}) && \text{in } \Omega; \\ u_i &= 0 && \text{on } \Gamma_D; \\ -K_i \nabla u_i \cdot \boldsymbol{\nu} &= 0 && \text{on } \Gamma_N; \\ u_i &= U_i && \text{on } \Gamma_0 \end{aligned}$$

for a given K_i, $i = 1, \ldots, n$. Search K_i such that $\int_{\Gamma_0} -K_i \nabla u_i \cdot \boldsymbol{\nu} = h_i$.

First result

Both solution methods give the existence of $(K_i, u_i) \in \mathbb{R}_+ \times V$ for $\tau < \tau_0$, with $V := \{\varphi \in H^1(\Omega);\ \varphi|_{\Gamma_D} = 0,\ \varphi|_{\Gamma_0} = const\}$.

Rothe functions



(a) (b)

Figure 1: Rothe's piecewise constant function $\overline{u}_n$ (a) and piecewise linear in time function u_n (b).

Variational formulation of IBVP

Find (K, u) such that

$$\begin{aligned} (\partial_t u, \varphi) + (K \nabla u, \nabla \varphi) + h \varphi|_{\Gamma_0} &= (f(u), \varphi), \quad \varphi \in V; \\ u|_{\Gamma_0} &= U. \end{aligned}$$

Second result (limit $n \to \infty$, i.e. $\tau \to 0$)

There exist a weak solution of the IBVP.

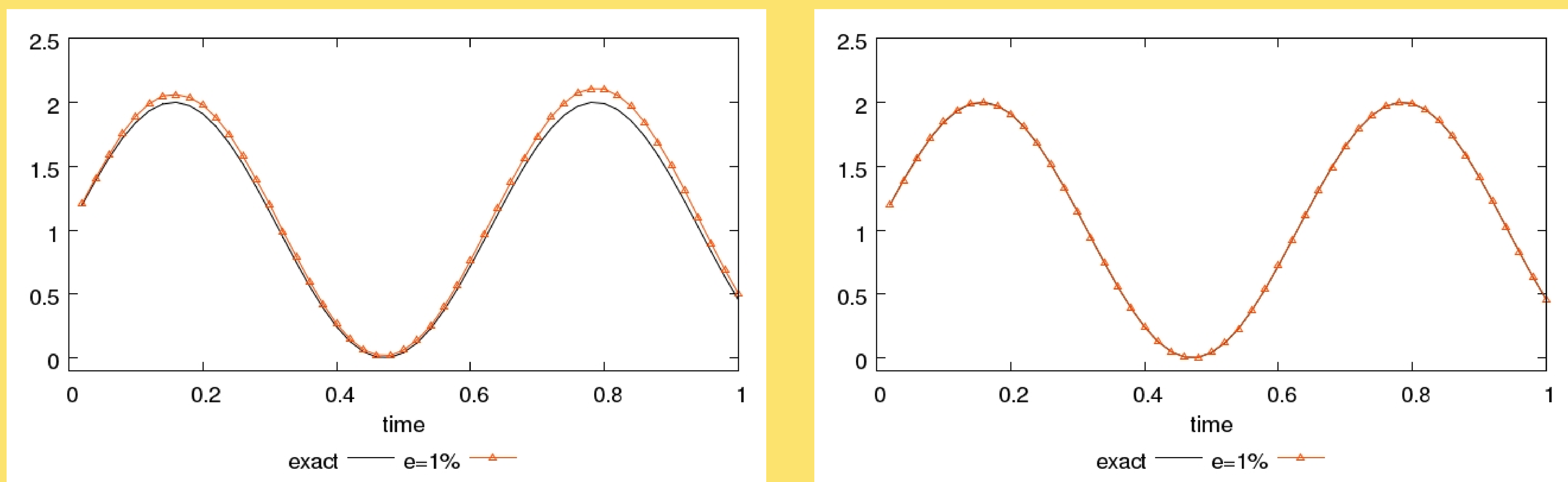
Numerical experiment

- ▷ $\Omega := (-\frac{1}{2}, 1) \times (-1, 1)$, $\Omega_0 := (-\frac{1}{2}, 0) \times (-1, 1)$, $T = 1$;
- ▷ $\Gamma_0 : x = -\frac{1}{2}$, $\Gamma_D : x = 1$, $\Gamma_N : y = -1$ and $y = 1$;
- ▷ $K(t, x, y) := \tilde{k}(t) \mathbf{1}_{\{x<0\}} + \frac{1}{2}$;
- ▷ The exact solution (for checking purposes):

$$K(t,x,y) := (1 + \sin(10t))\, \mathbf{1}_{\{x<0\}} + \frac{1}{2},$$
$$u(t,x,y) := (1+t) \sin\left(\frac{\pi}{2}(1-x)\right);$$

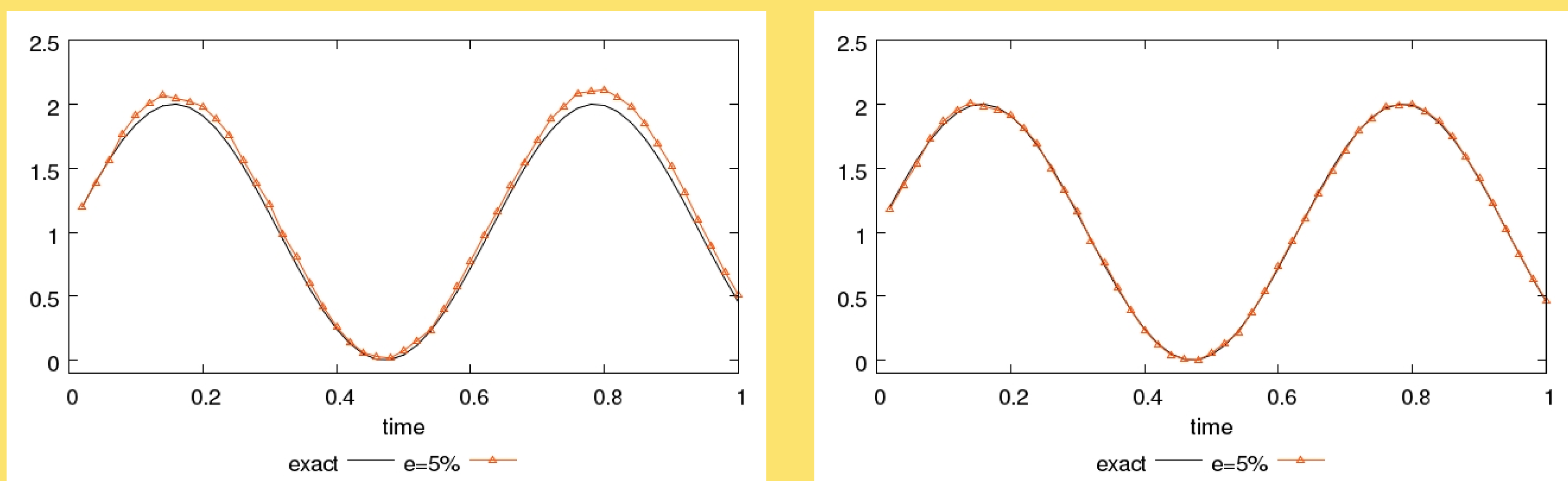
- ▷ Add noise with magnitude 1% and 5% to the additional condition $h(t) := \frac{\pi}{\sqrt{2}}(1+t)(1.5 + \sin(10t))$;
- ▷ The time interval is divided into small intervals of length $\tau = 0.02$;
- ▷ Lagrange P1- and P2-FEM: the space domain is divided into 144528 triangles;
- ▷ Recovery of $\tilde{k}(t)$: on each time-step t_i, $i = 1, \ldots, 50$, minimalize the functional (use the nonlinear conjugate gradient method)

$$J(\tilde{k}_i) := \left(\int_{\Gamma_0} (\tilde{k}_i + 0.5) \nabla u_i \cdot \boldsymbol{\nu} - h(t_i) \right)^2.$$



(a) (b)

Figure 2: Numerical value of $\tilde{k}_i$ using the P1-FEM (a) and P2-FEM (b) with noise $e = 1\%$; $i = 1, \ldots, 50$.



(a) (b)

Figure 3: Numerical value of $\tilde{k}_i$ using the P1-FEM (a) and P2-FEM (b) with noise $e = 5\%$; $i = 1, \ldots, 50$.

Further research

- ▷ Uniqueness of the solution?
- ▷ Non-linear differential operator: Richardson.

Acknowledgement

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