Inverse source problems in thermoelasticity

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Thermoelasticity

- ▷ Thermoelasticity is the change in the size and shape of a solid object as the temperature of that object fluctuates
- ▷ These interactions between the changes in the shape of an object and the fluctuations in the temperature are modeled by mathematical systems
- ▷ These so-called thermoelastic systems consist of two equations that are coupled: a parabolic (heat) equation and a vectorial hyperbolic equation for the displacement

Domain

- ▷ Bounded Lipschitz domain Ω in $\mathbb{R}^d$, $d \in \{1, 2, 3\}$
- ▷ Ω is occupied by an isotropic and homogeneous thermoelastic body

Thermoelastic systems

- ▷ The coupled thermoelastic system describing both the elastic and the thermal behaviours in Ω is given by

$$\begin{cases} \partial_{tt}\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla(\nabla\cdot\mathbf{u}) + \gamma\nabla\theta = \mathbf{p} & \text{in } \Omega\times(0,T), \\ \partial_t\theta - \rho\Delta\theta - k * \Delta\theta + \gamma\nabla\cdot\partial_t\mathbf{u} = h & \text{in } \Omega\times(0,T) \end{cases}$$

- ▷ $\mathbf{u} = (u_1, \ldots, u_d)^\top$ and θ denote respectively the displacement (in meters) and the temperature difference from the reference value (in Kelvin) of the solid elastic material at position $\mathbf{x}$ and time t
- ▷ The vector source $\mathbf{p}$ is a load (body force) vector and the source h is a heat source
- ▷ The Lamé parameters α and β, the coupling (absorbing) coefficient γ and the thermal coefficient ρ are assumed to be positive constants
- ▷ The sign '$*$' denotes the convolution product in time of a kernel k and a function θ, i.e.

$$(k*\theta)(\mathbf{x},t) := \int_0^t k(t-s)\theta(\mathbf{x},s)\,\mathrm{d}s, \qquad (\mathbf{x},t)\in\Omega\times(0,T)$$

Three types of thermoelasticity

Type-I thermoelasticity: $\rho \not\equiv 0, k \equiv 0$

Type-II thermoelasticity: $\rho \equiv 0, k \not\equiv 0$

Type-III thermoelasticity: $\rho \not\equiv 0, k \not\equiv 0$

Recovery of a solely space-dependent load vector source in thermoelastic systems [Van Bockstal and Slodička, 2015a]

- ▷ A solely space-dependent vector source $\mathbf{p}(\mathbf{x})$ is determined from a final in time measurement of the displacement
- ▷ An iterative method of Landweber-Fridman type (based on a sequence of well-posed problems) is proposed to recover the unknown source (thus not by minimizing a cost functional)
- ▷ The results are valid for all types of thermoelasticity if k is strongly positive definite
- ▷ Numerical experiments in 1D for type-I thermoelasticity

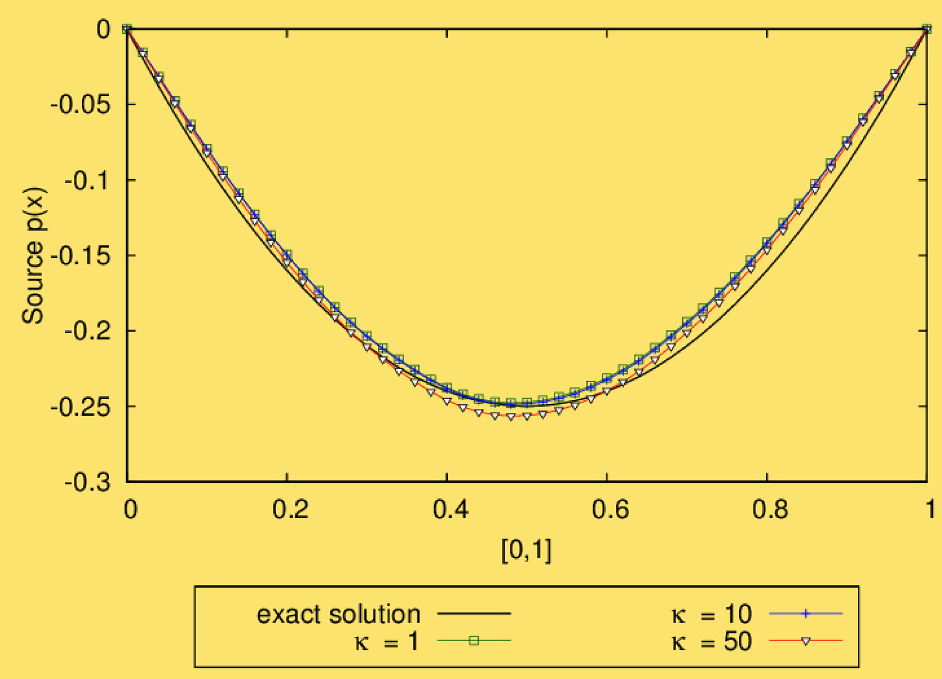

Figure 1 : The exact solution $p(x) = x(x-1)$ and the numerical solution for the source for different values of a relaxation parameter κ (noise with magnitude 1% on measurement)

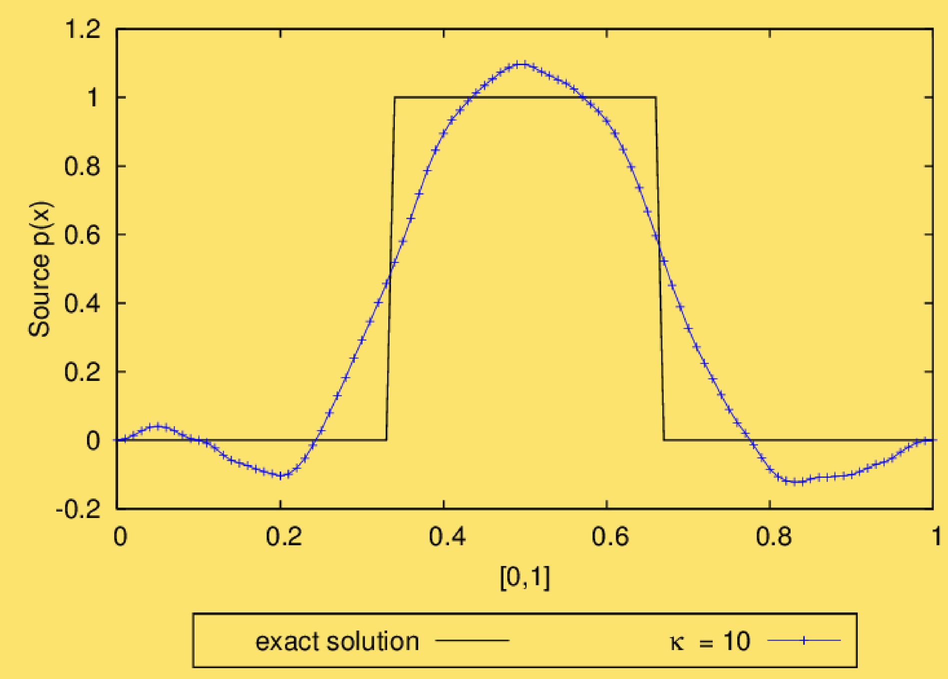

Figure 2 : The exact solution and the numerical solution (noise with magnitude 1% on measurement)

Recovery of a solely time-dependent heat source in 1D thermoelastic systems [Van Bockstal and Slodička, 2015b]

- ▷ A solely time-dependent heat source $h(t)$ is recovered from the averaged temperature when Ω is one-dimensional
- ▷ The inverse problem is recasted into a direct problem and the well-posedness of the problem is shown by using Rothe's method
- ▷ The results are valid for type-I and type-III thermoelasticity if $k \in C([0,T])$
- ▷ Numerical experiments in 1D for type-I thermoelasticity

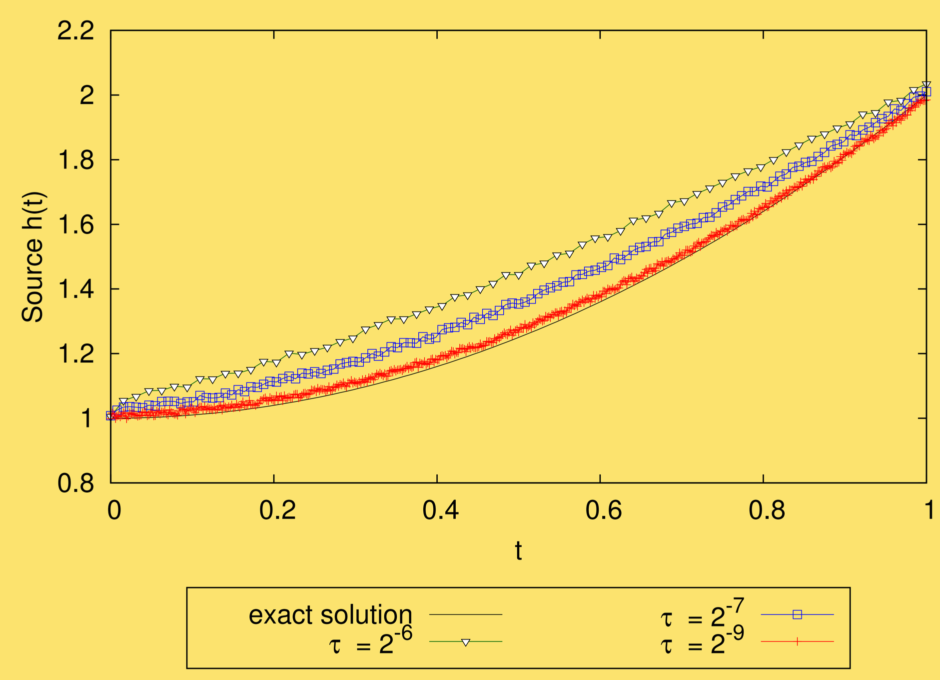

Figure 3 : Exact solution $h(t) = 1 + t^2$ and its numerical approximations for different values of the time discretization parameter τ (noise with magnitude 1% on derivative measurement)

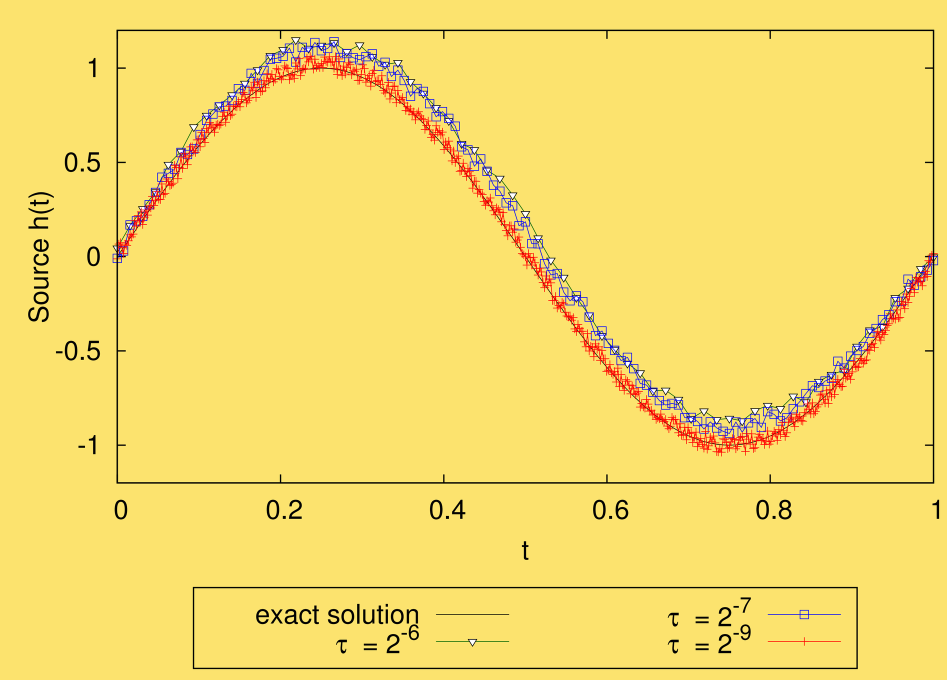

Figure 4 : Exact solution $h(t) = \sin(2\pi t)$ and numerical approximations for different values of the time discretization parameter τ (noise with magnitude 5% on derivative measurement)

Future research and references

Future research:
- ▷ More dimensional case, anisotropic materials
- ▷ Implementing relevant problems with correct physical parameters
- ▷ Other type of measurements

Van Bockstal, K. and Slodička, M. (2015a).
Recovery of a space-dependent vector source in thermoelastic systems.
Inverse Problems Sci. Eng., 23(6):956–968.

Van Bockstal, K. and Slodička, M. (2015b).
Recovery of a time-dependent heat source in one-dimensional thermoelasticity of type-III.
Inverse Problems Sci. Eng. (submitted).

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