# Determination of an unknown diffusion coefficient in a parabolic problem

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# Outline

Introduction
- Problem setting: application and mathematical model
- Time discretization

A single time-step
- Two solution methods
- Two auxiliary problems
- A priori estimates

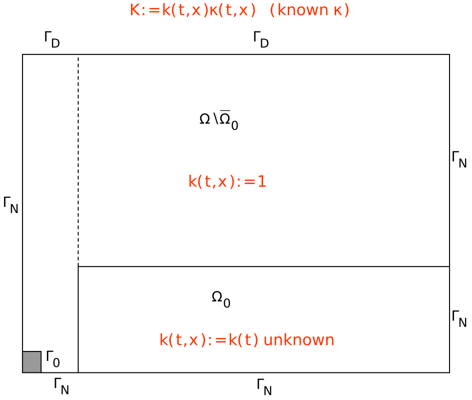
Convergence

Numerical experiment

Open questions

# Application: petroleum exploitation

- Spontaneous potential well-logging is an important technique to detect parameters (e.g. resistivity, diffusivity) of the formation in petroleum exploitation.
- The resistivity can depend on temperature and humidity in some geological formations. This makes the problem of the resistivity identification time-dependent.



# Mathematical model (IBVP)

Find $(K, u)$ such that ($T > 0$ fixed)

$$\begin{aligned}
\partial_t u - \nabla \cdot (K \nabla u) &= f(u) && \text{in } (0,T) \times \Omega; \\
u &= g^D && \text{in } (0,T) \times \Gamma_D; \\
-K \nabla u \cdot \boldsymbol{\nu} &= g^N && \text{in } (0,T) \times \Gamma_N; \\
u(0) &= u_0 && \text{in } \Omega;
\end{aligned}$$

with

$$\left\{ \begin{aligned}
\int_{\Gamma_0} -K \nabla u \cdot \boldsymbol{\nu} &= h(t) && \text{in } (0,T); \\
u &= U(t) && \text{on } (0,T) \times \Gamma_0.
\end{aligned} \right.$$

# Assumptions on the data

- $\overline{\Gamma}_D \cap \overline{\Gamma}_0 = \emptyset$ and $meas(\Gamma_0) > 0$;
- $0 < C_0 \leq k \leq C_1$ and $0 < D_0 \leq \kappa \leq D_1$;
- $U, h, \kappa \in C([0, T])$;
- $|f(x) - f(y)| \leq C|x - y|, \qquad \forall x, y$;
- $u_0 \in L_2(\Omega)$;
- $g^D = g^N = 0 \Rightarrow$ no loss of generality.

# Time discretization

- Rothe method: divide $[0, T]$ into $n \in \mathbb{N}$ equidistant subintervals $(t_{i-1}, t_i)$ for $t_i = i\tau$, where $\tau = T/n$ and for any function $z$
$$z_i := z(t_i), \qquad \delta z_i := \frac{z_i - z_{i-1}}{\tau};$$
- Recursive approximation scheme for $i = 1, \ldots, n$; $K_i = k_i \kappa_i$, with the unknown $(k_i, u_i)$ on each time-step

$$\begin{aligned}
\delta u_i - \nabla \cdot (K_i \nabla u_i) &= f(u_{i-1}) && \text{in } \Omega;\\
u_i &= 0 && \text{on } \Gamma_D;\\
-K_i \nabla u_i \cdot \boldsymbol{\nu} &= 0 && \text{on } \Gamma_N;\\
\int_{\Gamma_0} -K_i \nabla u_i \cdot \boldsymbol{\nu} &= h_i &&\\
u_i &= U_i && \text{on } \Gamma_0.
\end{aligned}$$

# First solution method

- Determine a solution of

$$\begin{aligned}\delta u_i - \nabla\cdot(K_i \nabla u_i) &= f(u_{i-1}) && \text{in } \Omega;\\ u_i &= 0 && \text{on } \Gamma_D;\\ -K_i\nabla u_i\cdot\boldsymbol{\nu} &= 0 && \text{on } \Gamma_N;\\ \int_{\Gamma_0} -K_i\nabla u_i\cdot\boldsymbol{\nu} &= h_i\end{aligned}$$

  for a given $k_i$ (recall that $K_i = k_i\kappa_i$).
- Search $k_i$ such that $u_i|_{\Gamma_0} = U_i$.

# Second solution method

- Solve

$$\begin{aligned} \delta u_i - \nabla \cdot (K_i \nabla u_i) &= f(u_{i-1}) && \text{in } \Omega; \\ u_i &= 0 && \text{on } \Gamma_D; \\ -K_i \nabla u_i \cdot \boldsymbol{\nu} &= 0 && \text{on } \Gamma_N; \\ u_i &= U_i && \text{on } \Gamma_0 \end{aligned}$$

  for a given $k_i$ (recall that $K_i = k_i \kappa_i$).
- Search $k_i$ such that $\int_{\Gamma_0} -K_i \nabla u_i \cdot \boldsymbol{\nu} = h_i$.

# First auxiliary problem

$$\frac{1}{\tau}(u,\varphi) + (k\kappa\nabla u, \nabla\varphi) + h\varphi|_{\Gamma_0} = (f,\varphi),$$

$$\forall\varphi \in V := \{\varphi \in H^1(\Omega);\ \varphi|_{\Gamma_D} = 0,\ \varphi|_{\Gamma_0} = const\}$$

- For all $k > 0$: unique weak solution $u_k \in H^1(\Omega)$;
- $\mathcal{T}(k) := u_k|_{\Gamma_0}$;
- $|\mathcal{T}(\alpha) - \mathcal{T}(\beta)| \leq C|\alpha - \beta|$ for $C_0 \leq k \leq C_1$ and $\tau < \tau_0$.

# Second auxiliary problem

$$\frac{1}{\tau}(u,\varphi) + (k\kappa\nabla u, \nabla\varphi) \quad = (f,\varphi)\,,$$

$$\forall\varphi \in \{\psi \in H^1(\Omega);\ \psi|_{\Gamma_0\cup\Gamma_D} = 0\} \subset V$$

- For all $k > 0$: unique weak solution $u_k \in H^1(\Omega)$;
- $\Psi(k) := \int_{\Gamma_0} -k\kappa\nabla u_k \cdot \boldsymbol{\nu}$;
- $|\Psi(\alpha) - \Psi(\beta)| \leq C|\alpha - \beta|$ for $C_0 \leq k \leq C_1$ and $\tau < \tau_0$.

# First result

Find $(k_i, u_i)$ such that

$$\begin{aligned} \delta u_i - \nabla\cdot(K_i\nabla u_i) &= f(u_{i-1}) && \text{in } \Omega; \\ u_i &= 0 && \text{on } \Gamma_D; \\ -K_i\nabla u_i\cdot\boldsymbol{\nu} &= 0 && \text{on } \Gamma_N; \\ \int_{\Gamma_0} -K_i\nabla u_i\cdot\boldsymbol{\nu} &= h_i && \\ u_i &= U_i && \text{on } \Gamma_0. \end{aligned} \tag{1}$$

## Lemma

*If $U(t) \in \mathcal{T}([C_0, C_1])\ \forall t \in [0, T]$ or $h(t) \in \Psi([C_0, C_1])\ \forall t \in [0, T]$, then there exist $\tau_0 > 0$ and $(k_i, u_i) \in \mathbb{R}_+ \times V$ which solves (1) for $\tau < \tau_0$.*

# A priori estimates

Variational formulation of IBVP on $t_i$:

$$\begin{aligned} (\delta u_i, \varphi) + (K_i \nabla u_i, \nabla \varphi) + h_i \varphi|_{\Gamma_0} &= (f(u_{i-1}), \varphi), \quad \varphi \in V; \\ u_i|_{\Gamma_0} &= U_i. \end{aligned}$$

## Lemma (Stability analysis $u_i$)

$$\max_{1 \le i \le n} \|u_i\|^2 + \sum_{i=1}^{n} \|u_i - u_{i-1}\|^2 + \sum_{i=1}^{n} \|\nabla u_i\|^2 \tau \le C$$

$$\sum_{i=1}^{n} \|\delta u_i\|_{V^*}^2 \tau \le C$$

# Rothe functions

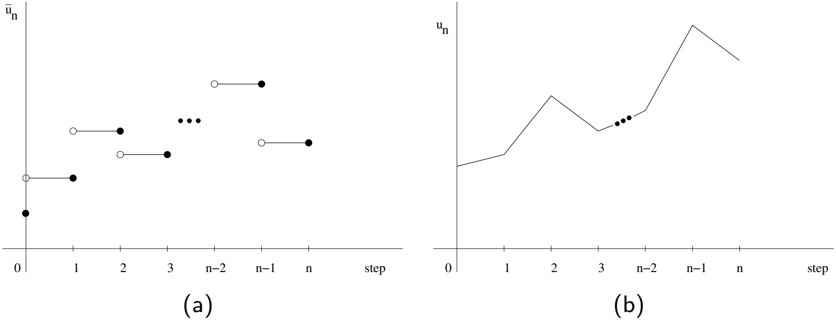



Figure: Rothe's piecewise constant function $\overline{u}_n$ (a) and Rothe's piecewise linear in time function $u_n$ (b).

- Similarly we define $\overline{K}_n, \overline{h}_n, \overline{U}_n$.

# Second result

- The variational formulation on $t_i$ can be rewritten as $(\varphi \in V)$

$$\begin{aligned} (\partial_t u_n, \varphi) + \left(\overline{K}_n \nabla \overline{u}_n, \nabla \varphi\right) + \overline{h}_n \varphi|_{\Gamma_0} &= \left(f(\overline{u}_n(t-\tau)), \varphi\right); \\ \overline{u}_n|_{\Gamma_0} &= \overline{U}_n. \end{aligned} \tag{2}$$

- The variational formulation of IBVP: find $(K, u)$ such that

$$\begin{aligned} (\partial_t u, \varphi) + (K \nabla u, \nabla \varphi) + h\varphi|_{\Gamma_0} &= (f(u), \varphi), \qquad \varphi \in V; \\ u|_{\Gamma_0} &= U. \end{aligned} \tag{3}$$

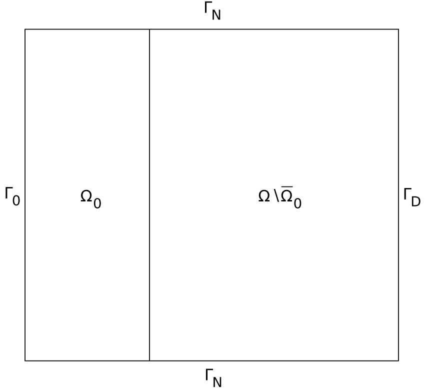
- The limit $\tau \to 0$ in (2) gives (3).

**Theorem**

*There exists a weak solution of the problem IBVP.*

# Numerical experiment: setting (1)

- $\Omega := \left(-\frac{1}{2}, 1\right) \times (-1, 1)$; $\Omega_0 := \left(-\frac{1}{2}, 0\right) \times (-1, 1)$;
- Time interval: $[0, 1]$.



# Numerical experiment: setting (2)

- $K(t,x,y) := \tilde{k}(t)\mathbf{1}_{\{x<0\}} + \frac{1}{2}$;
- Exact solution:

$$K(t,x,y) := (1+\sin(10t))\,\mathbf{1}_{\{x<0\}} + \frac{1}{2},$$
$$u(t,x,y) := (1+t)\sin\left(\frac{\pi}{2}(1-x)\right);$$

- Data: $g^D = g^N = 0$, $U(t) = \frac{1+t}{\sqrt{2}}$ and $u_0(x) = \sin\left(\frac{\pi}{2}(1-x)\right)$;
- Additional condition: $h(t) := \frac{\pi}{\sqrt{2}}(1+t)(1.5+\sin(10t))$;
- Add noise to this additional condition with given magnitude 1% and 5%.

# Numerical experiment: recovery of $\tilde{k}(t)$

- Time discretization: $\tau = 0.02$;
- Space discretization: regular triangulation of $\Omega$ consisting of 144528 triangles;
- Nonlinear conjugate gradient method: on each $t_i, i = 1, \dots, 50$, we minimalize

$$J(\tilde{k}_i) := \left( \int_{\Gamma_0} (\tilde{k}_i + 0.5) \nabla u_i \cdot \boldsymbol{\nu} - h(t_i) \right)^2; \quad \tilde{k}_1^{(0)} = 1;$$

- The resulting elliptic BVP at each time-step is solved numerically by the P1- and P2-FEM.

# Numerical experiment: results (1)

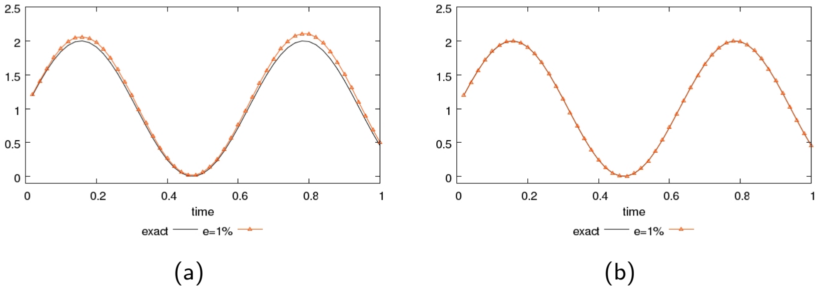



Figure: Numerical value of $\tilde{k}_i$ using the P1-FEM (a) and P2-FEM (b) with noise $e = 1\%$; $i = 1, \ldots, 50$.

# Numerical experiment: results (2)

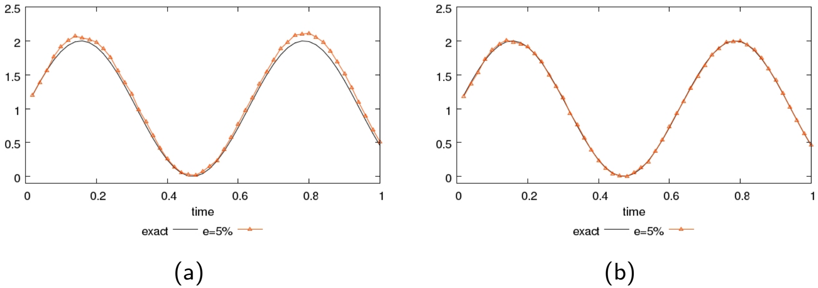



Figure: Numerical value of $\tilde{k}_i$ using the P1-FEM (a) and P2-FEM (b) with noise $e = 5\%$; $i = 1, \ldots, 50$.

# Numerical experiment: results (3)

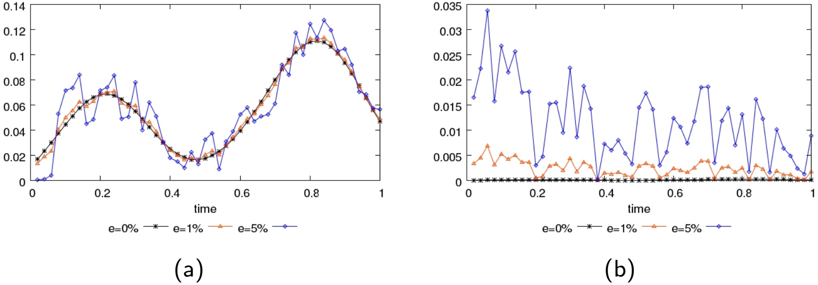



Figure: The absolute $\tilde{k}_i$-error using the P1-FEM (a) and the P2-FEM; $i = 1, \ldots, 50$.

# Open questions

- Uniqueness of the solution?
- Non-linear differential operator (Richardson):

$$\partial_t \theta(u) - \nabla \cdot (K \nabla u + \mathbf{a}(u)) \,.$$

# References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