UNIVERSITEIT GENT

FACULTY OF **ENGINEERING AND ARCHITECTURE**

Error analysis in the reconstruction of a convolution kernel in a semilinear parabolic problem with integral overdetermination

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Outline

- $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$: bounded domain with Lipschitz continuous boundary $\Gamma = \partial\Omega$, final time T.
- Determine the solution u and the convolution kernel $K(t)$ such that

$$\begin{cases} \partial_t u - \Delta u + K(t)h(\mathbf{x}, t) + K * u &= f(u, \nabla u), \qquad \text{in } \Omega \times [0, T], \\ -\nabla u \cdot \boldsymbol{\nu} &= g, \qquad \text{on } \Gamma \times [0, T], \\ u(\mathbf{x}, 0) &= u_0(\mathbf{x}), \qquad \text{in } \Omega \end{cases}$$

when an additional global measurement

$$\int_\Omega u(\mathbf{x}, t)\mathrm{d}\mathbf{x} = m(t)$$

is satisfied.

- The sign '$*$' denotes the convolution product

$$(K * u(\mathbf{x}))(t) := \int_0^t K(t-s)u(\mathbf{x}, s)\mathrm{d}s, \qquad (\mathbf{x}, t) \in \Omega \times [0, T].$$

- Measured problem

$$m'(t) + \int_\Gamma g + K(t)\int_\Omega h + K * m = \int_\Omega f(u, \nabla u). \qquad \text{(MP)}$$

- Variational problem for $\phi \in H^1(\Omega)$

$$(\partial_t u, \phi) + (\nabla u, \nabla \phi) + (g, \phi)_\Gamma + K(t)(h, \phi) + (K * u, \phi) = (f(u, \nabla u), \phi). \quad \text{(P)}$$

- [De Staelen and Slodička, 2014] proved existence and uniqueness of a solution:

Theorem (Existence and uniqueness)

Suppose f is bounded and Lipschitz continuous in all variables, $g \in C^1([0,T], L^2(\Gamma))$, $h \in C([0,T], H^1(\Omega)) \cap C^1([0,T], L^2(\Omega))$ and $\min_{t\in[0,T]} |(h(t), 1)| \geq \omega > 0$, $m \in C^2([0,T], \mathbb{R})$ and $u_0 \in H^2(\Omega)$. Then there exists a unique couple solutions $\langle u, K\rangle$ to (P)-(MP), where $u \in C([0,T], H^1(\Omega))$, $\partial_t u \in L^\infty([0,T], L^2(\Omega))$ and $K \in C([0,T])$, $K' \in L^2([0,T])$.

More details: talk on Thursday 27 June at 17h: kernel reconstruction in a semilinear parabolic problem with integral overdetermination.

- Rothe's method [Kačur, 1985]: divide $[0, T]$ into $n \in \mathbb{N}$ equidistant subintervals (t_{i-1}, t_i) for $t_i = i\tau$, where $\tau = T/n < 1$ and for any function z

$$z_i := z(t_i), \qquad \partial_t z(t_i) \approx \delta z_i := \tfrac{z_i - z_{i-1}}{\tau}.$$

- Time-discrete version of (P) at timestep t_i:

$$(\delta u_i, \phi) + (\nabla u_i, \nabla \phi) + (g_i, \phi)_\Gamma + K_i(h_i, \phi) + \sum_{k=1}^{i} (K_k u_{i-k}\tau, \phi) = (f_{i-1}, \phi) \quad (\mathrm{DP}i)$$

with $f_i = f(u_i, \nabla u_i)$.

- For given K_j, $j = 1, \ldots, i$, this is equivalent with solving $B(u_i, \phi) = F_i(\phi)$ with

$$B(u_i, \phi) = \frac{1}{\tau}(u_i, \phi) + (\nabla u_i, \nabla \phi),$$

$$F_i(\phi) = (f_{i-1}, \phi) - (g_i, \phi)_\Gamma - K_i(h_i, \phi) - \sum_{k=1}^{i} (K_k u_{i-k}\tau, \phi) + \frac{1}{\tau}(u_{i-1}, \phi).$$

- We obtain from (MP)

$$m_i' + (g_i, 1)_\Gamma + K_i(h_i, 1) + \sum_{k=1}^{i} K_k m_{i-k}\tau = (f_{i-1}, 1). \qquad \text{(DMP}i\text{)}$$

- On each time step t_i, we derive K_i from (DMPi) as follows

$$m_i' + (g_i, 1)_\Gamma + K_i(h_i, 1) + K_i m_0 \tau + \sum_{k=1}^{i-1} K_k m_{i-k}\tau = (f_{i-1}, 1). \qquad \text{(DMP}i\text{)}$$

- Use only solutions from previous timesteps!
- Then, derive u_i by solving $B(u_i, \phi) = F_i(\phi)$.

Algorithm: numerical scheme in pseudo code

input : $T > 0$, $n \in \mathbb{N}$ and functions f, g, h, m and u_0

output: kernel K and solution u at discrete time steps

$\tau \leftarrow T/n$;

$\theta \leftarrow [0 : \tau : T]$;

$\mathrm{K} \leftarrow \mathtt{zeros}(n+1)$;

$\mathrm{u} \leftarrow \mathtt{eval}(u_0, \theta)$;

$\mathrm{K}[0] \leftarrow \dfrac{1}{(h_0, 1)} \left((f(u_0, \nabla u_0), 1) - m_0' - (g_0, 1)_\Gamma \right)$;

for $i = 1$ **to** n **do**

$\quad \mathrm{K}[i] \leftarrow \dfrac{1}{(h_i, 1) + m_0 \tau} \left((f_{i-1}, 1) - (g_i, 1)_\Gamma - \sum_{k=1}^{i-1} K_k m_{i-k} \tau - m_i' \right)$;

$\quad \mathrm{u}[i] \leftarrow \mathtt{solveEP}(B(u_i, \phi) = F_i(\phi))$;

Rothe functions

- Piecewise constant and linear in time spline of the solutions $u_i, i = 1, \ldots, n$.

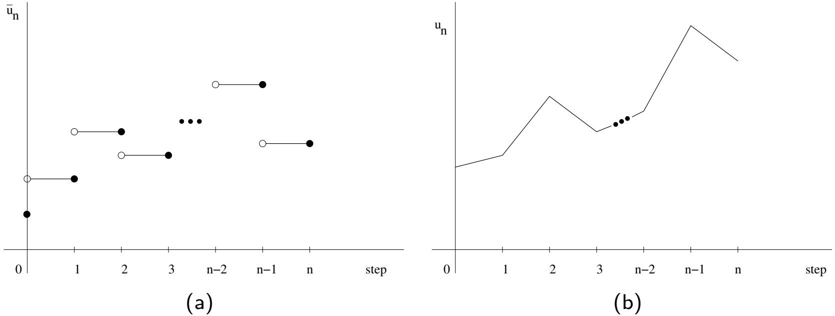


Figure : Rothe's piecewise constant function $\overline{u}_n$ (a) and Rothe's piecewise linear in time function u_n (b).

- Similarly, we define $\bar{K}_n$, $\bar{h}_n$, $\bar{g}_n$, $\bar{m}_n$ and $\overline{m'}_n$.

Using Rothe's functions, we can write (DPi) and (DMPi) on the whole time frame as

$$(\partial_t u_n, \phi) + (\nabla \bar{u}_n, \nabla \phi) + (\bar{g}_n, \phi)_\Gamma + \bar{K}_n(\bar{h}_n, \phi) + \sum_{k=1}^{\lfloor t \rfloor_\tau} (\bar{K}_n(t_k)\bar{u}_n(t-t_k)\tau, \phi) = (f(\bar{u}_n(t-\tau), \nabla \bar{u}_n(t-\tau)), \phi) \quad \text{(DP)}$$

where $\lfloor t \rfloor_\tau = i$ when $t \in (t_{i-1}, t_i]$, and

$$\overline{m'}_n + (\bar{g}_n, 1)_\Gamma + \bar{K}_n(\bar{h}_n, 1) + \sum_{k=1}^{\lfloor t \rfloor_\tau} \bar{K}_n(t_k)\bar{m}_n(t-t_k)\tau = (f(\bar{u}_n(t-\tau), \nabla \bar{u}_n(t-\tau)), 1). \quad \text{(DMP)}$$

Theorem (Error estimates [De Staelen et al., 2014])

Let the conditions of the existence theorem be fulfilled. Then, there exists a positive constant C, independent of the time step τ, such that

$$\max_{t\in[0,T]} |\bar{K}_n(t) - K(t)| \leq C\tau$$

and

$$\max_{t\in[0,T]} \|u_n(t) - u(t)\|^2 + \int_0^T \|\nabla u_n(t) - \nabla u(t)\|^2 \, \mathrm{d}t \leq C\tau^2.$$

Numerical experiment: setting

- $\Omega = [0, 1]$.
- The forward coupled problems in this procedure are discretized in time according to the backward Euler method with timestep $2^{-j}T, j = 5, \ldots, 9$.
- At each time-step, the resulting elliptic problems are solved numerically by the finite element method (FEM) using first order (P1-FEM) Lagrange polynomials for the space discretization. A fixed uniform mesh consisting of 50 intervals is used.
- The errors are respectively denoted by

$$E_K(\tau) = \max_{t \in [0,T]} |\bar{K}_n(t) - K_{\text{ex}}(t)| \approx \max_{0 \leqslant i \leqslant n} |\bar{K}_n(t_i) - K_{\text{ex}}(t_i)|$$

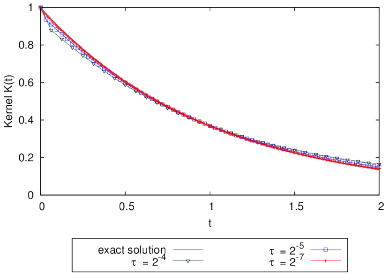
and

$$E_u(\tau) = \max_{t \in [0,T]} \|u_n(t) - u_{\text{ex}}(t)\|^2 \approx \max_{0 \leqslant i \leqslant n} \|u_n(t_i) - u_{\text{ex}}(t_i)\|^2 .$$

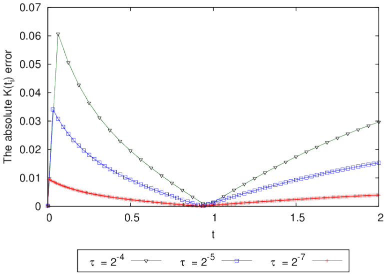
- Implementation: in FEniCS [Logg et al., 2012]

Experiment 1

$$T = 2, \quad f(r,s) = \sqrt{r^2 + \pi}, \quad m(t) = t^2 + t + 1,$$
$$u_{\mathrm{ex}}(x,t) = \left(t^2 + t + 1\right)\left(\cos\left(\pi x\right) + 1\right), \quad K_{\mathrm{ex}}(t) = \mathrm{e}^{-t}.$$



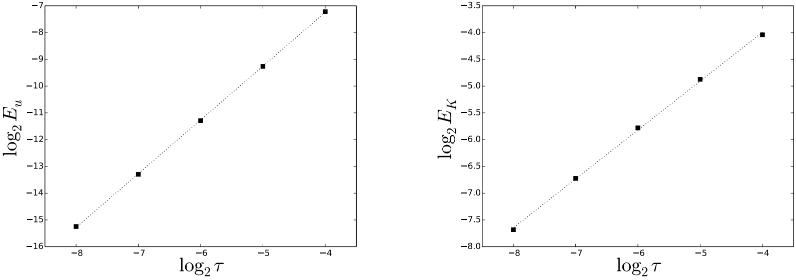
(a) exact solution and numerical solution



(b) absolute $K(t_i)$-error

Figure : Kernel reconstruction in Experiment 1.

Experiment 1



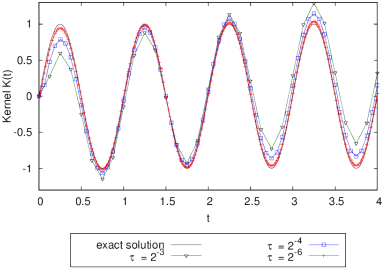
(a) error on u (b) error on K

Figure : Convergence rates for Experiment 1 on logarithmic scale.

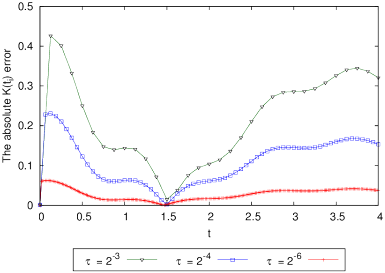
Linear regression lines: $\log_2 E_K = 0.9132 \log_2 \tau - 0.3412$ and $\log_2 E_u = 2.0086 \log_2 \tau + 0.7784$

Experiment 2

$$T = 4, \quad f(r,s) = \sqrt{r^2 + s^2 + \pi}, \quad m(t) = t^2 + t + 1,$$
$$u_{\rm ex}(x,t) = \left(t^2 + t + 1\right)\left(\cos\left(\pi\, x\right) + 1\right), \quad K_{\rm ex}(t) = \sin(2\pi t).$$

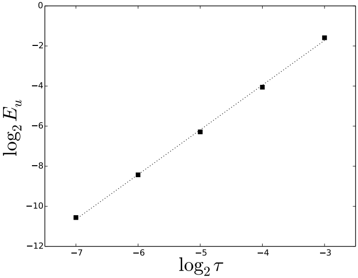


(a) exact solution and numerical solution

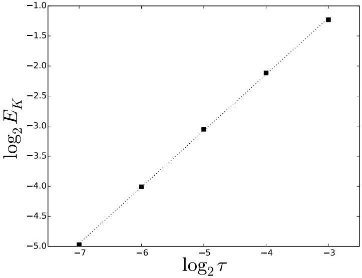


(b) absolute $K(t_i)$-error

Figure : Kernel reconstruction in Experiment 2.



(a) error on u



(b) error on K

Figure : Convergence rates for Experiment 2 on logarithmic scale.

Linear regression lines: $\log_2 E_K = 0.9378 \log_2 \tau + 1.6130$ and $\log_2 E_u = 2.2313 \log_2 \tau + 4.9715$.

Conclusion:

- A semilinear parabolic problem of second order with an unknown solely time-dependent convolution kernel is considered.
- A numerical scheme based on Backward Euler's method together with a time-discrete convolution is presented in oder to reconstruct the unknown convolution kernel based on an integral overdetermination.
- The convergence is of first order in time:

$$\max_{t\in[0,T]} |\bar{K}_n(t) - K_{\mathrm{ex}}(t)| \approx \mathcal{O}(\tau) \text{ and } \max_{t\in[0,T]} \|u_n(t) - u_{\mathrm{ex}}(t)\| \approx \mathcal{O}(\tau).$$

- Numerical experiments support the theoretically obtained results.

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