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Identification of a memory kernel in a nonlinear parabolic integro-differential problem

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Outline

- $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$: bounded domain with Lipschitz continuous boundary $\Gamma = \partial\Omega$, final time T.
- Determine the solution $u(x,t)$ and the convolution kernel $K(t)$ such that

$$\begin{cases} \partial_t u - \nabla \cdot (\nabla \beta(u)) + K * u & = & \text{`?'}, \quad \text{in } \Omega \times [0,T], \\ -\nabla \beta(u) \cdot \boldsymbol{\nu} & = & g, \quad \text{on } \Gamma \times [0,T], \\ u(\mathbf{x},0) & = & u_0(\mathbf{x}), \quad \text{in } \Omega, \end{cases}$$

when an additional global measurement

$$\int_\Omega u(\mathbf{x},t)\mathrm{d}\mathbf{x} = m(t), \qquad t \in [0,T]$$

is satisfied.
- The sign '$*$' denotes the convolution product

$$(K * u(\mathbf{x}))(t) := \int_0^t K(t-s)u(\mathbf{x},s)\mathrm{d}s, \qquad (\mathbf{x},t) \in \Omega \times [0,T].$$

- Such type of integro-differential problems arise in the theory of reactive contaminant transport [Delleur, 1999] and in the modelling of phenomena in viscoelasticity [MacCamy, 1977].
- [De Staelen and Slodička, 2015] studied the reconstruction of K based on the same measurement in the semilinear equation

$$\partial_t u - \Delta u + K(t)h + \int_0^t K(t-s)u(\mathbf{x},s)\ \mathrm{d}s = f(u,\nabla u).$$

- Main idea: measure the equation into space, i.e.

$$m'(t) + \int_\Gamma g(t) + K(t)\int_\Omega h(t) + (K*m)(t) = \int_\Omega f(u(t),\nabla u(t)).$$

The measured problem

$$m'(t) + \int_\Gamma g(t) + K(t) \int_\Omega h(t) + (K * m)(t) = \int_\Omega f(u(t), \nabla u(t)) \qquad \text{(MP)}$$

together with the variational formulation for $\phi \in H^1(\Omega)$

$$(\partial_t u, \phi) + (\nabla u, \nabla \phi) + (g, \phi)_\Gamma + K(t)(h, \phi) + (K * u, \phi) = (f(u, \nabla u), \phi) \quad \text{(P)}$$

represent the variational formulation of the inverse problem.

- The inverse problem is reformulated into a direct problem!
- Existence and uniqueness of a solution is proved using Rothe's method [Kačur, 1985].
- Uniform boundedness of K is crucial into the analysis (to obtain global in time solvability).

Uniform boundedness of K follows from (MP) and Grönwall's lemma:

$$|K(t)|\left|\int_\Omega h(t)\right| \leqslant \left|\int_\Omega f(u(t),\nabla u(t))\right| + |(K*m)(t)| + |m'(t)| + \left|\int_\Gamma g(t)\right|$$
$$\leqslant C + \int_0^t |K(s)| \, \mathrm{d}s$$

if $f:\mathbb{R}\to\mathbb{R}$ is uniformly bounded, $g \in \mathrm{C}\left([0,T],\mathrm{L}^2(\Gamma)\right)$ and $m \in \mathrm{C}^1([0,T])$.

$$\Rightarrow \max_{t\in[0,T]} |K(t)| \leqslant C \qquad \text{if } \min_{t\in[0,T]} |(h(t),1)| \geq \omega > 0.$$

- In this talk, the crucial $K(t)h$-term is skipped out of the PDE. What are the implications?

- Measure the problem:

$$m'(t) + \int_\Gamma g(t) + (K * m)(t) = \int_\Omega \text{`?'}.$$

- Idea: take the time derivative of this equation to obtain $K(t)$ seperately, i.e.

$$m''(t) + \int_\Gamma \partial_t g(t) + K(t)m(0) + (K * m')(t) = \partial_t \int_\Omega \text{`?'}.$$

- What is possible for '?' ?
- ~~$f(u)$~~.
- We make a safe choice for the right-hand side '?', i.e.

$$\int_0^t f(u(\cdot, s)) \, \mathrm{d}s.$$

- We have made the problem nonlinear by introducing the possible nonlinear function $\beta : \mathbb{R} \to \mathbb{R}$.

Determine the solution $u(x,t)$ and the convolution kernel $K(t)$ such that

$$\begin{cases} \partial_t u - \nabla \cdot (\nabla \beta(u)) + K * u &= \int_0^t f(u(\cdot,s))\ \mathrm{d}s + F, \quad \text{in } \Omega \times [0,T], \\ -\nabla \beta(u) \cdot \boldsymbol{\nu} &= g, \quad \text{on } \Gamma \times [0,T], \\ u(\mathbf{x},0) &= u_0(\mathbf{x}), \quad \text{in } \Omega, \\ \int_\Omega u(\mathbf{x},t)\mathrm{d}\mathbf{x} &= m(t). \end{cases}$$

The coupled direct variational problem is given by

$$m''(t) + K(t)m(0) + (K * m')(t) = (f(u(t)), 1) + (F'(t), 1) - (g'(t), 1)_\Gamma \quad \text{(MP2)}$$

and

$$(\partial_t u(t), \phi) + (\nabla \beta(u(t)), \nabla \phi) + ((K * u)(t), \phi) = \left(\int_0^t f(u(s)) \, \mathrm{d}s, \phi \right) + (F(t), \phi) - (g(t), \phi)_\Gamma , \quad \text{(P2)}$$

where $F'(t) := \partial_t F(t)$ and $g'(t) := \partial_t g(t)$.

- Rothe's method [Kačur, 1985]: divide $[0, T]$ into $n \in \mathbb{N}$ equidistant subintervals $(t_{i-1}, t_i]$ for $t_i = i\tau$, where $\tau = T/n < 1$ and for any function z

$$z_i \approx z(t_i), \qquad \partial_t z(t_i) \approx \delta z_i := \frac{z_i - z_{i-1}}{\tau}.$$

- Based on (P2) and (MP2), the following decoupled system for approximating the unknowns (K, u) at time t_i, $1 \leqslant i \leqslant n$, is proposed

$$(\delta u_i, \phi) + (\nabla \beta(u_i), \nabla \phi) + \left(\sum_{k=1}^{i} K_k u_{i-k} \tau, \phi \right) = \left(\sum_{k=0}^{i-1} f(u_k)\tau, \phi \right) + (F_i, \phi) - (g_i, \phi)_\Gamma \quad \text{(DP2}i\text{)}$$

and

$$m_i'' + K_i m(0) + \sum_{k=1}^{i} K_k m_{i-k}' \tau = (f(u_{i-1}), 1) + (F_i', 1) - (g_i', 1)_\Gamma . \quad \text{(DMP2}i\text{)}$$

We assume that

- $f : \mathbb{R} \to \mathbb{R}$ bounded,
- $\beta : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable and satisfies

$$\beta(0) = 0,$$
$$0 < \beta_0 \leqslant \beta'(s) \leqslant \beta_1, \quad \forall s \in \mathbb{R},$$

- $u_0 \in \mathrm{L}^2(\Omega)$,
- $g \in \mathrm{C}^1\left([0,T], \mathrm{L}^2(\Gamma)\right)$,
- $F \in \mathrm{C}^1\left([0,T], \mathrm{L}^2(\Omega)\right)$,
- $m \in \mathrm{C}^2\left([0,T]\right)$ with $m(0) \neq 0$.

and refer to these conditions as $(\star)$.

- Set $\tau_0 = \min\left\{1, \frac{|m(0)|}{2\,|m'(0)|}\right\}$. Then for any $\tau < \tau_0$, we get by the triangle inequality that

$$|m(0) + m'(0)\tau| \geqslant |m(0)| - |m'(0)|\,\tau \geqslant \frac{|m(0)|}{2} > 0.$$

For each $i \in \{1, \ldots, n\}$, the following recursive deduction can be made:

- Let $u_0, \ldots, u_{i-1} \in \mathrm{L}^2(\Omega)$ and $K_1, \ldots, K_{i-1} \in \mathbb{R}$ be given.
- Then, (DMP2i) implies the existence of a unique $K_i \in \mathbb{R}$ such that

$$K_i\,[m(0) + m'(0)\tau] = (f(u_{i-1}), 1) + (F_i', 1) - (g_i', 1)_\Gamma - \sum_{k=1}^{i-1} K_k m_{i-k}' \tau - m_i''.$$

- Monotone operator theory gives the existence of a unique solution $u_i \in \mathrm{H}^1(\Omega)$ to problem (DP2i) when the assumptions ($\star$) are fulfilled [Vainberg, 1973].

A priori estimates

Lemma

Let $(\star)$ *be satisfied. Then, there exists a positive constant* C *such that for any* $\tau < \tau_0$ *holds that*

$$\max_{i=1,\dots,n} |K_i| \leqslant C.$$

Lemma

Let $(\star)$ *be satisfied. Then there exist positive constants* C *such that for any* $\tau < \tau_0$ *holds that*

$$\max_{1\leqslant j\leqslant n} \|u_j\|^2 + \sum_{i=1}^{n} \|\nabla\beta(u_i)\|^2 \tau \leqslant C$$

and

$$\max_{0\leqslant j\leqslant n} \|\beta(u_j)\| \leqslant C \quad \text{and} \quad \sum_{i=1}^{n} \|u_i\|^2_{\mathrm{H}^1(\Omega)} \tau \leqslant C.$$

A priori estimates

Lemma

Let $(\star)$ *be satisfied and* $u_0 \in \mathrm{H}^1(\Omega)$. *Then there exist positive constants* C *such that for any* $\tau < \tau_0$ *holds that*

$$\sum_{i=1}^{n} \|\delta u_i\|^2 \tau + \max_{1\leqslant j\leqslant n} \|\nabla\beta(u_j)\|^2 + \sum_{i=1}^{j} \|\nabla\beta(u_i) - \nabla\beta(u_{i-1})\|^2 \leqslant C$$

and

$$\max_{1\leqslant j\leqslant n} \|u_j\|_{\mathrm{H}^1(\Omega)} \leqslant C \quad \text{and} \quad \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2 \leqslant C.$$

Rothe functions

- Piecewise constant and linear in time spline of the solutions $u_i, i = 1, \dots, n$.

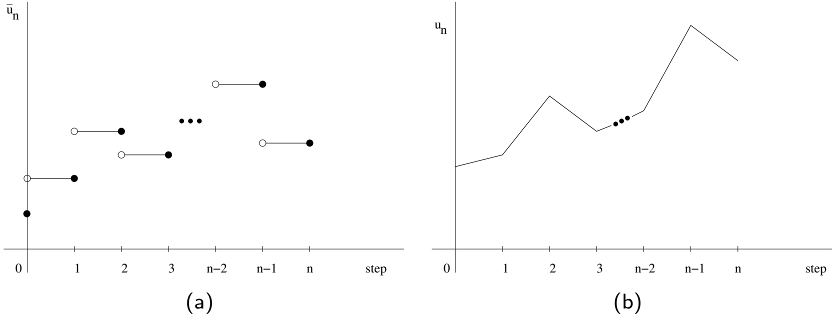


Figure : Rothe's piecewise constant function $\overline{u}_n$ (a) and Rothe's piecewise linear in time function u_n (b).

- Similarly, we define $\overline{K}_n$, $\overline{F}_n$, $\overline{F'}_n$, $\overline{g}_n$, $\overline{g'}_n$, $\overline{m'}_n$ and $\overline{m''}_n$.

Using these so-called Rothe's functions, (DP2i) and (DMP2i) can be rewritten on the whole time interval as ($\lceil t \rceil_\tau = i$ and $\lfloor t \rfloor_\tau = i - 1$ when $t \in (t_{i-1}, t_i]$)

$$(\partial_t u_n(t), \phi) + (\nabla\beta(\overline{u}_n(t)), \nabla\phi) + \left(\sum_{k=1}^{\lceil t \rceil_\tau} \overline{K}_n(t_k)\overline{u}_n(t - t_k)\tau, \phi\right) = \left(\sum_{k=0}^{\lfloor t \rfloor_\tau} f(\overline{u}_n(t_k))\tau, \phi\right) + (\overline{F}_n(t), \phi) - (\overline{g}_n(t), \phi)_\Gamma$$

and

$$\overline{m''}_n(t) + \overline{K}_n(t)m(0) + \sum_{k=1}^{\lceil t \rceil_\tau} \overline{K}_n(t_k)\overline{m'}_n(t - t_k)\tau = (f(\overline{u}_n(t - \tau)), 1) + \left(\overline{F'}_n(t), 1\right) - \left(\overline{g'}_n(t), 1\right)_\Gamma .$$

We want to pass to the limit $n \to \infty$ (term by term).

Theorem (Existence and uniqueness)

Suppose that the conditions ($\star$) *are fulfilled. Moreover, assume that* $u_0 \in \mathrm{H}^1(\Omega)$ *and let* $f : \mathbb{R} \to \mathbb{R}$ *be global Lipschitz continuous. Then, there exists a unique weak solution* $\langle K, u\rangle$ *to the problem* (P2)-(MP2), *where* $K \in \mathrm{L}^2(0,T)$ *and* $u \in \mathrm{C}\left([0,T], \mathrm{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0,T), \mathrm{H}^1(\Omega)\right)$ *with* $\partial_t u \in \mathrm{L}^2\left((0,T), \mathrm{L}^2(\Omega)\right)$.

Proof.

Uses Lemma 1.3.13 of [Kačur, 1985] (compactness argument: $\mathrm{H}^1(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}^2(\Omega)$). Note that by the a priori estimates holds that for every $t \in [0,T]$

$$\sum_{k=1}^{\lceil t \rceil_\tau} \overline{K}_n(t_k)\overline{u}_n(t-t_k)\tau = (\overline{K}_n * \overline{u}_n)(t) + \int_t^{\tau\lceil t \rceil_\tau} \overline{K}_n(s)\overline{u}_n(t-s)\,\mathrm{d}s = (\overline{K}_n * \overline{u}_n)(t) + \mathcal{O}(\tau)$$

and

$$\sum_{k=0}^{\lfloor t \rfloor_\tau} f(\overline{u}_n(t_k))\tau = f(u_0)\tau + \int_0^t f(\overline{u}_n(s))\,\mathrm{d}s - \int_{\tau\lfloor t \rfloor_\tau}^t f(\overline{u}_n(s))\,\mathrm{d}s = \int_0^t f(\overline{u}_n(s))\,\mathrm{d}s + \mathcal{O}(\tau).$$

□

- We have $u_n \to u$ in $\mathrm{C}\left([0,T],\mathrm{L}^2(\Omega)\right)$, $\overline{u}_n \to u$ in $\mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right)$.
- Only weak convergence of the Rothe functions $\overline{K}_n$ to K is proved up to now ($K_n \rightharpoonup K$). Extra assumptions are needed for the strong convergence.

Lemma

Let the assumptions $(\star)$ *be fulfilled and* $u_0 \in \mathrm{H}^1(\Omega)$. *Moreover, assume that* $\nabla\beta(u_0) \in \mathbf{H}(\mathbf{div};\Omega)$, $g \in \mathrm{C}^2\left([0,T],\mathrm{L}^2(\Gamma)\right)$, $F \in \mathrm{C}^2\left([0,T],\mathrm{L}^2(\Omega)\right)$, $m \in \mathrm{C}^3([0,T])$ *and* $f:\mathbb{R}\to\mathbb{R}$ *is global Lipschitz continuous. Then, there exist positive constants* C *and* τ_0 *such that for all* $\tau < \tau_0$ *holds that*

$$\sum_{i=1}^{n} |\delta K_i|^2 \tau \leqslant C.$$

By the Arzelà-Ascoli theorem [Rudin, 1987, Theorem 11.28], $\{K_n\}$ converges uniformly on $[0,T]$ to K, i.e. $K \in \mathrm{C}([0,T])$.

Error estimates (speed of convergence)

Theorem

Let the assumptions of the previous lemma be fulfilled. Then there exist positive constants C and τ_0 such that for all $\tau < \tau_0$ holds that

$$\int_0^T \left|\overline{K}_n(t) - K(t)\right|^2 \mathrm{d}t + \int_0^T \|\overline{u}_n(t) - u(t)\|^2 \leqslant C\tau^2.$$

- The convergence of the numerical approximations $(\overline{K}_n, \overline{u}_n)$ to the exact solution (K, u) is optimal in time.

Numerical experiment: setting

- $\Omega = [0, 1]$.
- The forward coupled problems in this procedure are discretized in time according to the backward Euler method with timestep $2^{-j}T, j = 2, \dots, 8$.
- At each time-step, the resulting elliptic problems are solved numerically by the finite element method (FEM) using first order (P1-FEM) Lagrange polynomials for the space discretization. A fixed uniform mesh consisting of 100 intervals is used.
- At each timestep, the nonlinearity $\nabla\beta(u_i)$ is approximated by $\beta'(u_{i-1})\nabla u_i$.
- The error on K is denoted by

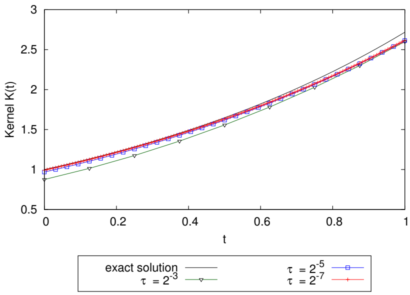
$$E_{K_{\mathrm{ex}}}(\tau) = \int_0^T |K_n(t) - K_{\mathrm{ex}}(t)|^2 \,\mathrm{d}t.$$

- Implementation: in FEniCS [Logg et al., 2012].

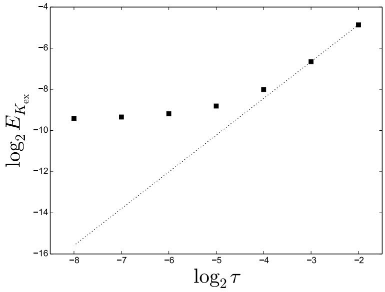
β linear

$$T = 1, \quad f(s) = \beta(s) = s + 1, \quad m(t) = 4/3\, t^2 + 4/3\, t + 4/3,$$
$$u_{\text{ex}}(x,t) = \left(1 + t + t^2\right)\left(1 + x^2\right), \quad K_{\text{ex}}(t) = \exp(t)$$

$$\log_2 E_{K_{\text{ex}}} = 1.7875 \log_2 \tau - 1.2878$$



(a) Kernel reconstruction



(b) Error $E_{K_{\text{ex}}}(\tau)$

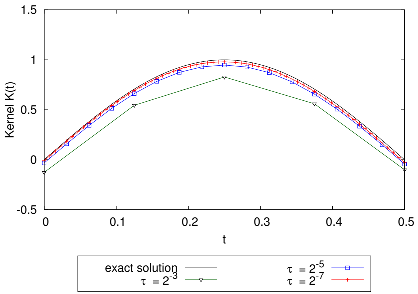
β nonlinear

$$T = \frac{1}{2}, \quad f(s) = s + 5, \beta(s) = s^2 + s,$$

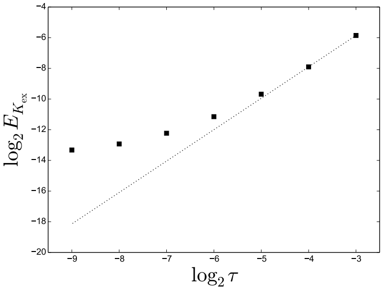
$$m(t) = \frac{\pi t^3 + \pi t^2 + 2t^3 + \pi t + 2t^2 + \pi + 2t + 2}{\pi},$$

$$u_{\mathrm{ex}}(x,t) = \left(1 + t + t^2 + t^3\right)\left(1 + \sin(\pi x)\right), \quad K_{\mathrm{ex}}(t) = \sin(2\pi t)$$

$$\log_2 E_{K_{\mathrm{ex}}} = 2.0478 \log_2 \tau + 0.2911$$



(c) Kernel reconstruction



(d) Error $E_{K_{\mathrm{ex}}}(\tau)$.

Conclusion:

- A nonlinear parabolic problem of second order with an unknown solely time-dependent convolution kernel is considered.
- A numerical scheme based on Backward Euler's method together with a time-discrete convolution is presented in oder to reconstruct the unknown convolution kernel based on an integral overdetermination.
- The convergence is of first order in time:

$$\left\|\bar{K}_n(t) - K_{\mathrm{ex}}(t)\right\|_{\mathrm{L}^2((0,T))} \approx \mathcal{O}(\tau).$$

- Numerical experiments support the theoretically obtained results.

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