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The identification of a space-dependent load source in anisotropic thermoelastic systems

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Outline

- $\Omega \subset \mathbb{R}^d, d \in \{1,2,3\}$: isotropic and homogeneous thermoelastic body
- $\Gamma = \partial\Omega$: Lipschitz continuous boundary
- T: final time
- Coupled thermoelastic system [Muñoz Rivera and Qin, 2002]: specific formulas are used in the study of thermoelasticity to describe how objects change in shape (displacement vector $\mathbf{u}$) with changes in temperature θ from the reference value $T_0 > 0$ (in Kelvin)

$$\begin{cases} \rho\partial_{tt}\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla\left(\nabla\cdot\mathbf{u}\right) + \gamma\nabla\theta &= \mathbf{p} \qquad \text{in } \Omega\times(0,T) \\ \rho C_s\partial_t\theta - \kappa\Delta\theta - K*\Delta\theta + T_0\gamma\nabla\cdot\partial_t\mathbf{u} &= h \qquad \text{in } \Omega\times(0,T) \end{cases}$$

- $\mathbf{p}$: load (body force) vector; h: heat source
- The Lamé parameters α and β, the mass density ρ, the specific heat C_s, the coupling (absorbing) coefficient γ and the thermal coefficient κ are assumed to be positive constants
- The sign '$*$' denotes the convolution product

$$\left(K*\theta\right)(\mathbf{x},t) := \int_0^t K(t-s)\theta(\mathbf{x},s)\mathrm{d}s, \qquad (\mathbf{x},t)\in\Omega\times(0,T)$$

Types of thermoelasticity

$$\begin{cases} \rho\partial_{tt}\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla(\nabla\cdot\mathbf{u}) + \gamma\nabla\theta & = \mathbf{p} & \text{in } \Omega\times(0,T); \\ \rho C_s\partial_t\theta - \kappa\Delta\theta - K*\Delta\theta + T_0\gamma\nabla\cdot\partial_t\mathbf{u} & = h & \text{in } \Omega\times(0,T); \\ \mathbf{u}(\mathbf{x},0) = \overline{\mathbf{u}}_0(\mathbf{x}), \quad \partial_t\mathbf{u}(\mathbf{x},0) = \overline{\mathbf{u}}_1(\mathbf{x}), \quad \theta(\mathbf{x},0) & = \overline{\theta}_0(\mathbf{x}) & \text{in } \Omega \end{cases}$$

Three types of thermoelasticity:

- type-I: $K = 0$ and $\kappa \neq 0$:

$$\rho C_s\partial_t\theta - \kappa\Delta\theta + T_0\gamma\nabla\cdot\partial_t\mathbf{u} = h$$

- type-II: $K \neq 0$ and $\kappa = 0$:

$$\rho C_s\partial_t\theta - K*\Delta\theta + T_0\gamma\nabla\cdot\partial_t\mathbf{u} = h$$

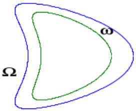
- type-III: $K \neq 0$ and $\kappa \neq 0$:

$$\rho C_s\partial_t\theta - \kappa\Delta\theta - K*\Delta\theta + T_0\gamma\nabla\cdot\partial_t\mathbf{u} = h$$

Inverse source problems for (an-)isotropic thermoelasticity are studied

- [Bellassoued and Yamamoto, 2011] investigated an inverse heat source problem for type-I thermoelasticity: they determine $h(\mathbf{x})$ by measuring

$$\mathbf{u}_{|\omega\times(0,T)} \text{ and } \theta(\cdot, t_0),$$



where ω is a subdomain of Ω such that $\Gamma \subset \partial\omega$ and $t_0 \in (0, T)$

- [Wu and Liu, 2012] studied an inverse source problem of determining $\mathbf{p}(\mathbf{x})$ for type-II thermoelasticity from a displacement measurement

$$\mathbf{u}_{|\omega\times(0,T)}$$

- Using a Carleman estimate, a Hölder stability for the inverse source problem is proved in both contributions, which implies the uniqueness of a solution to the inverse source problem
- Gap: no numerical scheme is provided to recover the unknown source

Problem (A)

Can we find a unique $\mathbf{p}(\mathbf{x})$ *and/or* $h(\mathbf{x})$ *from the additional final in time measurements*

$$\mathbf{u}(\mathbf{x}, T) = \boldsymbol{\xi}_T(\mathbf{x}) \text{ and/or } \theta(\mathbf{x}, T) = \zeta_T(\mathbf{x})$$

for all types of thermoelasticity and can we provide a numerical scheme?

Goal: The way of retrieving the unknown source is not by the minimization of a certain cost functional (which is typical for IPs), but by using an alternative technique

Solution (Problem (A))

Up to now, using our approach, it is possible to recover $\mathbf{p}(\mathbf{x})$ *uniquely for all types of thermoelasticity from the additional final in time measurement (the condition of final overdetermination)*

$$\mathbf{u}(\mathbf{x}, T) = \boldsymbol{\xi}_T(\mathbf{x}),$$

in the presence of a damping term $\mathbf{g}\left(\partial_t \mathbf{u}\right)$ *in the hyperbolic equation of the thermoelastic system, i.e.*

$$\begin{cases} \rho \partial_{tt}\mathbf{u} + \mathbf{g}\left(\partial_t \mathbf{u}\right) - \alpha \Delta \mathbf{u} - \beta \nabla\left(\nabla \cdot \mathbf{u}\right) + \gamma \nabla \theta &= \mathbf{p}(\mathbf{x}) \quad \text{in } \Omega \times (0,T); \\ \rho C_s \partial_t \theta - \kappa \Delta \theta - K * \Delta \theta + T_0 \gamma \nabla \cdot \partial_t \mathbf{u} &= 0 \quad \text{in } \Omega \times (0,T); \\ \mathbf{u}(\mathbf{x},t) &= \mathbf{0} \quad \text{on } \Gamma \times (0,T); \\ \theta(\mathbf{x},t) &= 0 \quad \text{on } \Gamma \times (0,T); \\ \mathbf{u}(\mathbf{x},0) = \partial_t \mathbf{u}(\mathbf{x},0) = \mathbf{0}, \quad \theta(\mathbf{x},0) &= 0 \quad \text{in } \Omega, \end{cases}$$

- A damping term in thermoelastic systems is also considered in [Qin, 2008, Chapter 9], [Kirane and Tatar, 2001], [Oliveira and Charão, 2008],...

See: Van Bockstal, K. and Slodička, M. *Recovery of a space-dependent vector source in thermoelastic systems.*
Inverse Problems in Science and Engineering, 2015, 23, 956–968

The results can be extended to anisotropic thermoelastic systems

$$\begin{cases} \varrho(\mathbf{x})\partial_{tt}\mathbf{u} + \mathbf{g}\left(\partial_t \mathbf{u}\right) + \mathcal{L}^{\mathrm{e}}\mathbf{u} + \mathrm{div}(\mathbb{B}(\mathbf{x})\theta) = \mathbf{p}(\mathbf{x}) + \mathbf{r}, & (\mathbf{x}, t) \in \Omega \times (0, T), \\ \varrho(\mathbf{x})C_s(\mathbf{x})\partial_t\theta - \nabla \cdot (\mathbb{K}(\mathbf{x})\nabla\theta) - (K * \Delta\theta) + T_0\mathbb{B}(\mathbf{x}) : \nabla\partial_t\mathbf{u} = h, & (\mathbf{x}, t) \in \Omega \times (0, T), \\ \mathbf{u}(\mathbf{x}, t) = \mathbf{0}, & (\mathbf{x}, t) \in \Gamma \times (0, T), \\ \theta(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in \Gamma \times (0, T), \end{cases}$$

together with the initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{0}, \quad \partial_t \mathbf{u}(\mathbf{x}, 0) = \mathbf{0}, \quad \theta(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \Omega.$$

As before, the goal is to determine $\mathbf{p}(\mathbf{x})$ from

$$\mathbf{u}_T(\mathbf{x}) := \mathbf{u}(\mathbf{x}, T) = \boldsymbol{\xi}_T(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

See: Van Bockstal, K. and Marin, L. *Recovery of a space-dependent vector source in anisotropic thermoelastic systems.*
Computer Methods in Applied Mechanics and Engineering, 2017, 321, 269–293

- Overview results (in both papers):
  - A variational approach is used, which implies uniqueness for all types of thermoelasticity if $\mathbf{g} : \mathbb{R}^d \mapsto \mathbb{R}^d$ is strictly monotone increasing and K is strongly positive definite
  - if $\mathbf{g}$ is linear (i.e. $\mathbf{g} = g\mathbf{I}$ with $g > 0$), then
    - A stable iterative algorithm is proposed to recover the unknown vector source $\mathbf{p}$ by extending the iterative procedure of [Johansson and Lesnic, 2007] for the heat equation to thermoelastic systems, but without using an adjoint problem
    - It is possible to consider the case of non-homogeneous Dirichlet boundary conditions and initial conditions
    - Also additional given source terms can be considered
- In the following: more details are given for isotropic thermoelasticity of type-III

Theorem (Well-posedness of the direct problem (given general $\mathbf{p}$))

Assume that $\mathbf{p} : (0, T] \to \mathbf{L}^2(\Omega)$ *belong to* $\mathrm{L}^2\left((0, T), \mathbf{L}^2(\Omega)\right)$, $\overline{\mathbf{u}}_0(\mathbf{x}) \in \mathbf{H}^1(\Omega)$, $\overline{\mathbf{u}}_1(\mathbf{x}) \in \mathbf{L}^2(\Omega)$ *and* $\overline{\theta}_0 \in \mathrm{H}^1(\Omega)$. *Assume that any of the following conditions holds for the kernel* $K : (0, T] \to \mathbb{R}$*:*

(i) $K'(t) \not\equiv 0$ *and* $(-1)^j K^{(j)}(t) \geqslant 0, t > 0, j = 0, 1, 2,$ *i.e.* K *is strongly positive definite;*

(ii) $K \in \mathrm{L}^1(0, T)$ *s.t.* $\displaystyle\int_0^T |K(t)|\,\mathrm{d}t \leqslant \kappa$;

(iii) $\exists\ C > 0$ *s.t.* $\displaystyle\max_{t\in[0,T]} |K(t)| \leqslant C$.

Then, the variational problem has a unique solution $(\mathbf{u}, \theta)$ *such that*
$\mathbf{u} \in \mathrm{C}\left([0, T], \mathbf{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0, T), \mathbf{H}_0^1(\Omega)\right)$, $\partial_t \mathbf{u} \in \mathrm{C}\left([0, T], \mathbf{L}^2(\Omega)\right)$, $\partial_{tt}\mathbf{u} \in \mathrm{L}^2\left((0, T), \mathbf{H}_0^1(\Omega)^*\right)$,
$\theta \in \mathrm{C}\left([0, T], \mathbf{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0, T), \mathrm{H}_0^1(\Omega)\right)$ *and* $\partial_t\theta \in \mathrm{L}^2\left((0, T), \mathrm{H}_0^1(\Omega)^*\right)$.

Moreover, when $\overline{\mathbf{u}}_0(\mathbf{x}) = \mathbf{0}$, $\overline{\mathbf{u}}_1(\mathbf{x}) = \mathbf{0}$, $\overline{\theta}_0 = 0$, $h = 0$ *and* $\mathbf{p} = \mathbf{p}(\mathbf{x})$, *the following estimate holds*

$$\max_{t\in[0,T]} \left\{ \|\partial_t \mathbf{u}(t)\|^2 + \|\mathbf{u}(t)\|^2_{\mathbf{H}_0^1(\Omega)} + \|\theta(t)\|^2 \right\} + \int_0^T \|\nabla\theta(t)\|^2\,\mathrm{d}t \leqslant C\,\|\mathbf{p}\|^2 .$$

- See [Lions and Magenes, 1972] and [Van Bockstal and Marin, 2017, Theorem 4.1]

Coupled variational formulation: find $\langle \mathbf{u}(t), \theta(t), \mathbf{p}\rangle \in \mathbf{H}_0^1(\Omega) \times \mathrm{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)$ such that $\mathbf{u}(\mathbf{x}, T) = \boldsymbol{\xi}_T(\mathbf{x})$ and

$$\rho\left(\partial_{tt}\mathbf{u}, \boldsymbol{\varphi}\right) + \left(\mathbf{g}\left(\partial_t \mathbf{u}\right), \boldsymbol{\varphi}\right) + \alpha\left(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}\right) + \beta\left(\nabla \cdot \mathbf{u}, \nabla \cdot \boldsymbol{\varphi}\right) + \gamma\left(\nabla \theta, \boldsymbol{\varphi}\right) = \left(\mathbf{p}, \boldsymbol{\varphi}\right),$$
$$\rho C_s\left(\partial_t \theta, \psi\right) + \kappa\left(\nabla \theta, \nabla \psi\right) + \left(k * \nabla \theta, \nabla \psi\right) - \gamma T_0\left(\partial_t \mathbf{u}, \nabla \psi\right) = 0,$$

for all $\boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega)$ and $\psi \in \mathrm{H}_0^1(\Omega)$ and a.a. $t \in (0, T]$.

Theorem (Uniqueness)

Let $\langle \mathbf{u}_1, \theta_1, \mathbf{p}_1\rangle$ and $\langle \mathbf{u}_2, \theta_2, \mathbf{p}_2\rangle$ satisfy the thermoelastic system. Set $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2$ and $\theta = \theta_1 - \theta_2$ such that $\mathbf{u}(\mathbf{x}, 0) = \mathbf{0}$, $\mathbf{u}(\mathbf{x}, T) = \mathbf{0}$, $\partial_t \mathbf{u}(\mathbf{x}, 0) = \mathbf{0}$ and $\theta(\mathbf{x}, 0) = 0$. Then $\mathbf{p} = \mathbf{0}$ a.e. in Ω and $\langle \mathbf{u}, \theta\rangle = \langle \mathbf{0}, 0\rangle$ a.e. in $\Omega \times (0, T)$.

- Subtract, equation by equation, the variational formulation corresponding with the different solutions
- We want to add up both resulting equation such that the mixed term is cancelled out
- A good choice of the test functions is needed:

$$\boldsymbol{\varphi} = \partial_t \mathbf{u}(t) \quad \text{and} \quad \psi = \frac{\theta(t)}{T_0}$$

- Another trick: integrate in time over $(0, T)$ such that

$$\int_\Omega \int_0^T \mathbf{p}(\mathbf{x}) \cdot \partial_t \mathbf{u}(\mathbf{x}, t) \mathrm{d}t = \int_\Omega [\mathbf{p}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}, T) - \mathbf{p}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}, 0)] = 0$$

- We obtain that

$$\frac{\rho}{2} \|\partial_t \mathbf{u}(T)\|^2 + \underbrace{\int_0^T (\mathbf{g}(\partial_t \mathbf{u_1}) - \mathbf{g}(\partial_t \mathbf{u_2}), \partial_t \mathbf{u}_1 - \partial_t \mathbf{u}_2)}_{?}$$
$$+ \frac{\rho C_s}{2T_0} \|\theta(T)\|^2 + \frac{\kappa}{T_0} \int_0^T \|\nabla\theta\|^2 + \underbrace{\frac{1}{T_0} \int_0^T (K * \nabla\theta, \nabla\theta)}_{?} = 0$$

- We make distinction based on the different assumptions on K

Uniqueness for a Positive Definite Convolution Kernel I

- Assume that the twice differentiable function $K : (0, T] \to \mathbb{R}$ satisfies

$$K'(t) \not\equiv 0 \quad \text{and} \quad (-1)^j K^{(j)}(t) \geqslant 0, \quad t > 0, \quad j = 0, 1, 2,$$

  i.e. K is strongly positive definite

$$\int_0^T \phi(t)(K * \phi)(t)\mathrm{d}t \geqslant C_0 \int_0^T (K * \phi)^2 (t)\mathrm{d}t, \qquad \forall T > 0, \forall \phi \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$$

- This implies

$$\int_0^T \left(\mathbf{g}\left(\partial_t \mathbf{u_1}\right) - \mathbf{g}\left(\partial_t \mathbf{u_2}\right), \partial_t \mathbf{u}_1 - \partial_t \mathbf{u}_2\right) + \int_0^T \|\nabla\theta\|^2 \leqslant 0$$

Uniqueness for a Positive Definite Convolution Kernel II

- Assume $\mathbf{g}$ componentwise strictly monotone increasing. Then $\mathbf{u}_t = \mathbf{0}$ a.e. in $\Omega \times (0, T)$. Therefore,

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{0} \quad \Rightarrow \quad \mathbf{u}(\mathbf{x}, t) = \mathbf{0} \text{ a.e. in } \Omega \times (0, T)$$

- $\theta = 0$ on $\partial\Omega \Rightarrow \theta = 0$ a.e. in $\Omega \times (0, T)$
- This implies that

$$(\mathbf{p}, \boldsymbol{\varphi}) = 0, \qquad \forall \boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega).$$

  From this, we conclude that $\mathbf{p} = \mathbf{0}$ in $\mathbf{L}^2(\Omega)$
- Examples:
  - E.g. $K(t) = t^{-\alpha}$, $t \in (0, T]$, with $0 < \alpha < 1$ (singular kernel)
  - E.g. $K(t) = \exp(-t)$, $t \in [0, T]$

Uniqueness for $K \in \mathrm{L}^1(0,T)$ s.t. $\int_0^T |K(t)|\,\mathrm{d}t \leqslant \kappa$.

Young's inequality for convolutions:

$$\|f * g\|_r \leqslant \|f\|_p \|g\|_q\,, \quad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1, \quad 1 \leqslant p,q,r \leqslant \infty. \tag{1}$$

Applying this inequality, one obtains

$$\begin{aligned}
&\left|\int_0^T ((K*\nabla\theta)(t),\nabla\theta(t))\,\mathrm{d}t\right| = \left|\int_\Omega\int_0^T (K*\nabla\theta)(\mathbf{x},t)\nabla\theta(\mathbf{x},t)\,\mathrm{d}t\mathrm{d}\mathbf{x}\right| \\
&\leqslant \int_\Omega \left|\int_0^T (K*\nabla\theta)(\mathbf{x},t)\nabla\theta(\mathbf{x},t)\,\mathrm{d}t\right|\mathrm{d}\mathbf{x} \\
&\leqslant \int_\Omega \sqrt{\int_0^T (K*\nabla\theta)^2(\mathbf{x},t)\,\mathrm{d}t}\sqrt{\int_0^T \nabla\theta(\mathbf{x},t)^2\,\mathrm{d}t}\;\mathrm{d}\mathbf{x} \\
&\overset{(1)}{\leqslant} \int_\Omega \left(\int_0^T |K(t)|\,\mathrm{d}t\right)\sqrt{\int_0^T \nabla\theta(\mathbf{x},t)^2\,\mathrm{d}t}\sqrt{\int_0^T \nabla\theta(\mathbf{x},t)^2\,\mathrm{d}t}\;\mathrm{d}\mathbf{x} \\
&\leqslant \left(\int_0^T |K(t)|\,\mathrm{d}t\right)\int_0^T \|\nabla\theta(t)\|^2\,\mathrm{d}t,
\end{aligned}$$

Uniqueness for a bounded convolution kernel

$$
\begin{aligned}
&\left|\int_0^T \left(\int_0^t K(t-s)\nabla\theta(s)\mathrm{d}s, \nabla\theta(t)\right)\mathrm{d}t\right| \\
&\leqslant C_\varepsilon \int_0^T \left\|\int_0^t K(t-s)\nabla\theta(s)\mathrm{d}s\right\|^2 \mathrm{d}t + \varepsilon \int_0^T \|\nabla\theta(t)\|^2 \mathrm{d}t \\
&\leqslant C_\varepsilon \int_0^T \left(\int_0^t |K(t-s)|\,\|\nabla\theta(s)\|\,\mathrm{d}s\right)^2 \mathrm{d}t + \varepsilon \int_0^T \|\nabla\theta(t)\|^2 \mathrm{d}t \\
&\leqslant C_\varepsilon \int_0^T \left(\int_0^t |K(t-s)|^2 \mathrm{d}s\right)\left(\int_0^t \|\nabla\theta(s)\|^2 \mathrm{d}s\right)\mathrm{d}t + \varepsilon \int_0^T \|\nabla\theta(t)\|^2 \mathrm{d}t \\
&\leqslant C_\varepsilon \int_0^T \left(\int_0^t \|\nabla\theta(s)\|^2 \mathrm{d}s\right)\mathrm{d}t + \varepsilon \int_0^T \|\nabla\theta(t)\|^2 \mathrm{d}t.
\end{aligned}
$$

Fixing ε sufficiently small and applying Grönwall's lemma implies that $\mathbf{u} = \mathbf{p} = \mathbf{0}$ and $\theta = 0$.

Algorithm for finding the source term if g is linear

(i) Choose an initial guess $\mathbf{p}_0 \in \mathbf{L}^2(\Omega)$. Let $\langle \mathbf{u}_0, \theta_0 \rangle$ be the solution to the thermoelastic system with $\mathbf{p} = \mathbf{p}_0$

(ii) Assume that $\mathbf{p}_k$ and $\langle \mathbf{u}_k, \theta_k \rangle$ have been constructed. Let $\langle \mathbf{w}_k, \eta_k \rangle$ solve the thermoelastic system with $\mathbf{p}(\mathbf{x}) = \mathbf{u}_k(\mathbf{x}, T) - \boldsymbol{\xi}_T(\mathbf{x})$

(iii) Define

$$\mathbf{p}_{k+1}(\mathbf{x}) = \mathbf{p}_k(\mathbf{x}) - \omega \mathbf{w}_k(\mathbf{x}, T), \quad \mathbf{x} \in \Omega$$

where $\omega > 0$ (relaxation parameter), and let $\langle \mathbf{u}_{k+1}, \theta_{k+1} \rangle$ solve the thermoelastic system with $\mathbf{p} = \mathbf{p}_{k+1}$

(iv) The procedure continues by repeating steps (ii) and (iii) until a desired level of accuracy is achieved (see next slide)

- This is a Landweber-Fridmann iteration scheme [Fridman, 1956].
- The proof of convergence can be found in [Van Bockstal and Slodička, 2015, Theorem 3.3] for isotropic materials and in [Van Bockstal and Marin, 2017, Theorem 4.2]

Stopping criterion

- Morozov's discrepancy principle is used [Morozov, 1966]
- The case is considered when there is some error in the additional measurement, i.e.

$$\|\boldsymbol{\xi}_T - \boldsymbol{\xi}_T^e\| \leqslant e,$$

where $e(\tilde{e})$ depends on the noise level with magnitude $\tilde{e} > 0$
- The solutions $\mathbf{p}_k^e, \mathbf{u}_k^e$ and θ_k^e at iteration k are obtained by using the algorithm
- The discrepancy principle suggests to finish the iterations at the smallest index $k = k(e, \omega)$ for which

$$E_{k,\mathbf{u}_T} = \left\|\mathbf{u}_k^e(\cdot, T) - \widetilde{\boldsymbol{\xi}}_T^e\right\| \leqslant e$$

Numerical experiment: setting

- 1D linear model for isotropic type-I ($K = 0$) and type-III thermoelasticity is considered
- $\Omega = [0, 1]$, $T = 1$
- copper alloy: shear modulus $G = 4.8 \times 10^{10}\,\mathrm{N}/\mathrm{m}^2$, Poisson's ratio $\nu = 0.34$, $\alpha_T = 16.5 \times 10^{-6}\,1/\mathrm{K}$, $\kappa = 401\,\mathrm{W}/\mathrm{mK}$, $\rho = 8960\,\mathrm{kg}/\mathrm{m}^3$ and $C_s = 385\,\mathrm{J}/\mathrm{kgK}$
- $g = 2 \times 10^8$, $T_0 = 293\mathrm{K}$
- $\alpha = \mu$, $\beta = \mu + \lambda$ with $\lambda = \dfrac{2\nu G}{1 - 2\nu}$ and $\mu = G$
- Three choices for the convolution kernel are made, namely $K = 0$, $K = \exp(-t)$ and $K = 1/\sqrt{t}$

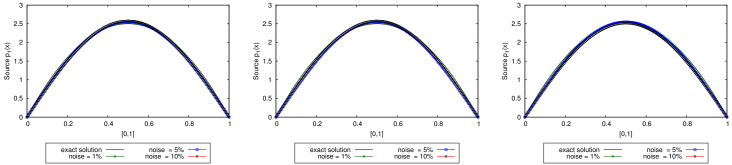
Numerical experiment: setting

- The forward coupled problems in this procedure are discretized in time according to the backward Euler method with timestep 0.0005
- At each time-step, the resulting elliptic coupled problems are solved numerically by the finite element method (FEM) using first order (P1-FEM) Lagrange polynomials for the space discretization. A fixed uniform mesh consisting of 200 intervals is used
- The finite element library DOLFIN [Logg and Wells, 2010, Logg et al., 2012b] from the FEniCS project [Logg et al., 2012a] is used

Exact solution

$$u(x,t) = (1+t)^2 x(x-1)^2 \quad \text{and} \quad \theta(x,t) = (1+t)x(1-x)^2$$

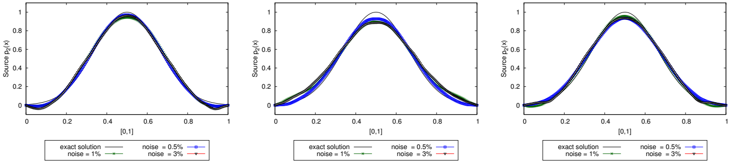
$$p_1(x) = 10x(1-x)$$



(a) (b) (c)

Figure: The exact source p_1 and its corresponding numerical solution, retrieved using various levels of noise in the additional measurement, for various convolution kernels, namely (a) $K = 0$, (b) $K = \exp(-t)$, and (c) $K = 1/\sqrt{t}$. The relaxation parameter $\omega = 10$.

$$p_2(x) = \exp\left(-20(x-0.5)^2\right)$$



(a) (b) (c)

Figure: The exact source p_2 and its corresponding numerical solution, retrieved using various levels of noise in the additional measurement, for various convolution kernels, namely (a) $K = 0$, (b) $K = \exp(-t)$, and (c) $K = 1/\sqrt{t}$. The relaxation parameter $\omega = 10$.

Table: The stopping iteration number $\tilde{k} = k(e(\tilde{e}), 10)$ and the CPU time (mins), obtained for the experiments with the unknown sources p_1 and p_2.

$\tilde{e}$	p_1 1%		p_1 5%		p_1 10%		p_2 0.5%		p_2 1%		p_2 3%	
	$\tilde{k}$	time	$\tilde{k}$	time	$\tilde{k}$	time	$\tilde{k}$	time	$\tilde{k}$	time	$\tilde{k}$	time
$K = 0$	136	94.7	11	8.2	9	6.3	387	327.4	386	327.2	172	60.7
$K = \exp(-t)$	133	138.7	9	10.4	9	9.9	503	538.1	321	416.2	177	111.2
$K = 1/\sqrt{t}$	142	144.3	10	11	8	8.9	491	532.6	390	468.4	206	183.4

Following experiments:

$$p_3(x) = \begin{cases} 0 & 0 \leqslant x \leqslant \frac{1}{3} \\ 6x - 2 & \frac{1}{3} \leqslant x \leqslant \frac{1}{2} \\ 4 - 6x & \frac{1}{2} \leqslant x \leqslant \frac{2}{3} \\ 0 & \frac{2}{3} \leqslant x \leqslant 1 \end{cases}, \qquad p_4(x) = \begin{cases} x(0.5 - x)(1 - x) & 0 \leqslant x \leqslant \frac{1}{2} \\ x(x - 0.5)(1 - x) & \frac{1}{2} \leqslant x \leqslant 1 \end{cases},$$

$$p_5(x) = \begin{cases} 0 & 0 \leqslant x < \frac{1}{3} \\ 1 & \frac{1}{3} \leqslant x \leqslant \frac{2}{3} \\ 0 & \frac{2}{3} < x \leqslant 1 \end{cases}, \qquad p_6(x) = 10x(x-1)^2$$

Results of numerical experiments

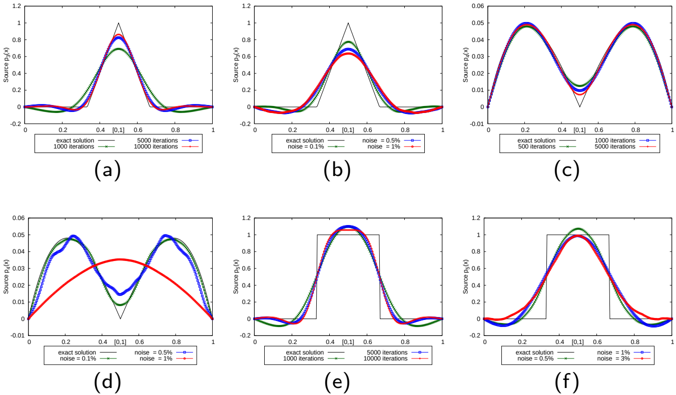
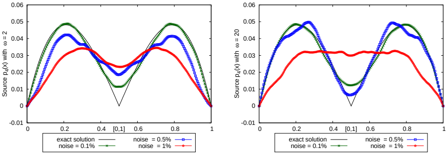


Figure: The exact sources p_3, p_4 and p_5 and its numerical approximations for $\tilde{e} = 0\%$ (a,c,e) and for different noise levels (b,d,f). The relaxation parameter $\omega = 10$.

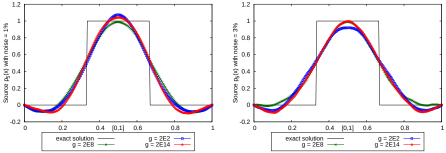
Other relaxation parameter



(a) (b)

Figure: The exact source p_4 and its numerical approximations for $\omega = 2$ (a) and for $\omega = 20$ (b).

- The results for small noise are similar to the results obtained when $\omega = 10$



(a) (b)

Figure: The exact source p_5 and its numerical approximations for $\tilde{e} = 1\%$ (a) and for $\tilde{e} = 3\%$ (b) for different values of g. The relaxation parameter $\omega = 10$.

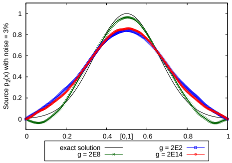
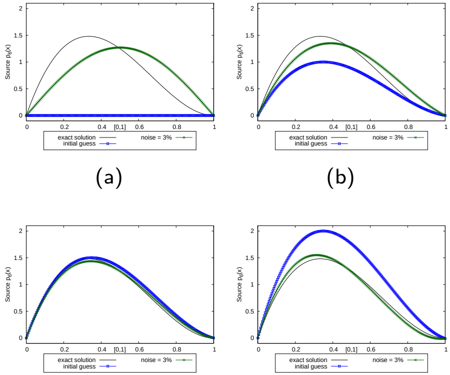


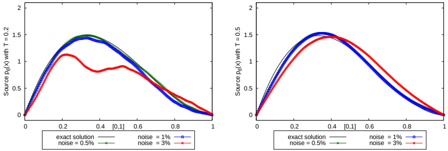
Figure: The exact source p_2 and its numerical approximations for $\tilde{e} = 3\%$ for different values of g. The relaxation parameter $\omega = 10$.



(a) (b)

(c) (d)

Figure: The non-symmetric exact source p_6 and its numerical approximations (using $\tilde{e} = 3\%$) for different initial guesses: 0 (a), $6.44x - 12.27x^2 + 5.83x^3$ (b), $9.68x - 18.46x^2 + 8.78x^3$ (c) and $12.88x - 24.54x^2 + 11.65x^3$ (d). The relaxation parameter $\omega = 10$.



(a) (b)

Figure: The non-symmetric exact source p_6 and its numerical approximations for $T = 0.2$ (a) and $T = 0.5$ (b). The relaxation parameter $\omega = 10$.

Conclusion

- It is possible to recover uniquely an unknown vector source in all types of damped thermoelastic systems when an additional final in time measurement of the displacement is measured
- A numerical algorithm in a linear case gives accurate shape recovery
- The algorithm is sensitive to the amount of noise added to the data
- There is a certain limitation of the method with respect to the recovery of non-symmetric sources

Future research

- More numerical experiments (e.g. influence of the parameter g on the results)
- Testing different stopping criteria (up to now, no better results)
- What if $\mathbf{g}$ is nonlinear?
- Other inverse problems for thermoelasticity, e.g. the recovery of time-dependent sources, convolution kernel
- Goal: with numerical scheme!

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