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Recovery of space-dependent sources in thermoelastic systems

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Outline

Reconstruction of a heat source: problem setting

- $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$: bounded domain with Lipschitz continuous boundary $\Gamma = \partial\Omega$, final time T
- The temperature u, heat source f and initial temperature distribution u_0 satisfy

$$\begin{cases} \partial_t u - \Delta u = f(\mathbf{x}) & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \Gamma \times (0, T), \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega. \end{cases}$$

- The forward problem is well-posed
- Suppose that $f(\mathbf{x})$ is unknown

Reconstruction of a heat source: inverse problem

- Consider the inverse problem

$$\begin{cases} \partial_t u - \Delta u = f(\mathbf{x}) & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \Gamma \times (0,T), \\ u(\mathbf{x},0) = u_0(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega, \\ u(\mathbf{x},T) = \psi_T(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega. \end{cases}$$

- Define the operator

$$A : L^2(\Omega) \to L^2(\Omega) : f \mapsto Af = u(\cdot, T).$$

Then the inverse problem is equivalent with solving the operator equation

$$Af = \psi_T.$$

- A is completely continuous $\Rightarrow$ this inverse problem is ill-posed
- Existence and uniqueness of the solution to this inverse problem is studied by [Cannon, 1968], [Rundell and Colton, 1980], [Prilepko and Solov'ev, 1987], [Solov'ev, 1989], [Isakov, 1990],...

Reconstruction of a heat source: how to solve?

- By minimizing the functional

$$J(f) = \|Af - \psi_T\|^2$$

  [Hasanov, 2007, Johansson and Lesnic, 2007a]
- [Johansson and Lesnic, 2007b] proposed an iterative procedure for finding the source based on a sequence of well-posed direct problems given the final overdetermination ψ_T
- Both approaches made use of an adjoint problem
- Extension of the previous results to a hyperbolic-parabolic coupled thermoelastic systems without using an adjoint problem

- $\Omega \subset \mathbb{R}^d$ is an isotropic and homogeneous thermoelastic body, $d \geqslant 1$
- $\mathbf{u} = (u_1, \ldots, u_d)$ denotes the displacement at the location $\mathbf{x}$ and the time t
- θ is the temperature difference from the reference value (in Kelvin) of the solid elastic material
- Assuming null surface displacement on the whole boundary, the classical thermoelastic system is given by

$$\begin{cases} \partial_{tt}\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla\left(\nabla\cdot\mathbf{u}\right) + \gamma\nabla\theta &= \mathbf{f}(\mathbf{x}) \quad \text{in } \Omega\times(0,T);\\ \partial_t\theta - \rho\Delta\theta - k\star\Delta\theta + \gamma\nabla\cdot\partial_t\mathbf{u} &= h(\mathbf{x}) \quad \text{in } \Omega\times(0,T);\\ \mathbf{u}(\mathbf{x},t) &= \mathbf{0} \quad \text{on } \Gamma\times(0,T);\\ \theta(\mathbf{x},t) &= 0 \quad \text{on } \Gamma\times(0,T); \end{cases}$$

with initial conditions:

$$\mathbf{u}(\mathbf{x},0) = \overline{\mathbf{u}}_0(\mathbf{x}), \quad \partial_t\mathbf{u}(\mathbf{x},0) = \overline{\mathbf{u}}_1(\mathbf{x}), \quad \theta(\mathbf{x},0) = \overline{\theta}_0(\mathbf{x}), \quad \mathbf{x}\in\Omega$$

- The sign '$\star$' denotes the convolution product

$$\left(k\star\theta\right)(\mathbf{x},t) := \int_0^t k(t-s)\theta(\mathbf{x},s)\mathrm{d}s, \qquad (\mathbf{x},t)\in\Omega\times(0,T)$$

$$\begin{cases} \partial_{tt}\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla(\nabla\cdot\mathbf{u}) + \gamma\nabla\theta &= \mathbf{f}(\mathbf{x}) \quad \text{in } \Omega\times(0,T);\\ \partial_t\theta - \rho\Delta\theta - k\star\Delta\theta + \gamma\nabla\cdot\partial_t\mathbf{u} &= h(\mathbf{x}) \quad \text{in } \Omega\times(0,T);\\ \mathbf{u}(\mathbf{x},t) &= \mathbf{0} \quad \text{on } \Gamma\times(0,T);\\ \theta(\mathbf{x},t) &= 0 \quad \text{on } \Gamma\times(0,T);\\ \mathbf{u}(\mathbf{x},0)=\overline{\mathbf{u}}_0(\mathbf{x}),\quad \partial_t\mathbf{u}(\mathbf{x},0)=\overline{\mathbf{u}}_1(\mathbf{x}),\quad \theta(\mathbf{x},0) &= \overline{\theta}_0(\mathbf{x}) \quad \text{in } \Omega \end{cases}$$

Three types of thermoelasticity:

- type-I: $k = 0$ and $\rho \neq 0$ in the parabolic equation:

$$\partial_t\theta - \rho\Delta\theta + \gamma\nabla\cdot\partial_t\mathbf{u} = h(\mathbf{x})$$

- type-II: $k \neq 0$ and $\rho = 0$ in the parabolic equation:

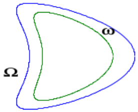
$$\partial_t\theta - k\star\Delta\theta + \gamma\nabla\cdot\partial_t\mathbf{u} = h(\mathbf{x})$$

- type-III: $k \neq 0$ and $\rho \neq 0$ in the parabolic equation:

$$\partial_t\theta - \rho\Delta\theta - k\star\Delta\theta + \gamma\nabla\cdot\partial_t\mathbf{u} = h(\mathbf{x})$$

- [Bellassoued and Yamamoto, 2011] investigated an inverse heat source problem for type-I thermoelasticity: they determine $h(\mathbf{x})$ by measuring

$$\boldsymbol{u}_{|\omega\times(0,T)} \text{ and } \theta(\cdot, t_0),$$



  where ω is a subdomain of Ω such that $\Gamma \subset \partial\omega$ and $t_0 \in (0, T)$
- [Wu and Liu, 2012] studied an inverse source problem of determining $\mathbf{f}(\mathbf{x})$ for type-II thermoelasticity from a displacement measurement

$$\mathbf{u}_{|\omega\times(0,T)}$$

- Using a Carleman estimate, a Hölder stability for the inverse source problem is proved in both contributions, which implies the uniqueness of the inverse source problem
- No numerical scheme is provided to recover the unknown source

Problem

Can we find a unique $\mathbf{f}(\mathbf{x})$ *and/or* $h(\mathbf{x})$ *from the additional final time measurements*

$$\mathbf{u}(\mathbf{x}, T) = \xi_T(\mathbf{x}) \text{ and/or } \theta(\mathbf{x}, T) = \zeta_T(\mathbf{x})$$

for all types of thermoelasticity and can we provide a numerical scheme?

Solution

Using our approach, it is possible to recover $\mathbf{f}(\mathbf{x})$ *uniquely from the additional final time measurement*

$$\mathbf{u}(\mathbf{x}, T) = \xi_T(\mathbf{x}),$$

in the presence of a damping term $\mathbf{g}(\partial_t \mathbf{u}) = (g_1(\partial_t \mathbf{u}), g_2(\partial_t \mathbf{u}), g_3(\partial_t \mathbf{u}))$ *in the hyperbolic equation of the thermoelastic system*

- We use a variational approach which implies uniqueness for all types of thermoelasticity
- We propose a stable iterative algorithm to recover the unknown vector source $\mathbf{f}$ by extending the iterative procedure of [Johansson and Lesnic, 2007b] to thermoelastic systems

Find $\langle \mathbf{u}, \theta, \mathbf{f} \rangle$ such that

$$\begin{cases} \partial_{tt}\mathbf{u} + \mathbf{g}\left(\partial_t \mathbf{u}\right) - \alpha\Delta\mathbf{u} - \beta\nabla\left(\nabla\cdot\mathbf{u}\right) + \gamma\nabla\theta & = \mathbf{f}(\mathbf{x}) & \text{in } \Omega\times(0,T); \\ \partial_t\theta - \rho\Delta\theta - k\star\Delta\theta + \gamma\nabla\cdot\partial_t\mathbf{u} & = 0 & \text{in } \Omega\times(0,T); \\ \mathbf{u}(\mathbf{x},t) & = \mathbf{0} & \text{on } \Gamma\times(0,T); \\ \theta(\mathbf{x},t) & = 0 & \text{on } \Gamma\times(0,T); \\ \mathbf{u}(\mathbf{x},0) = \overline{\mathbf{u}}_0(\mathbf{x}), \quad \partial_t\mathbf{u}(\mathbf{x},0) = \overline{\mathbf{u}}_1(\mathbf{x}), \quad \theta(\mathbf{x},0) & = \overline{\theta}_0(\mathbf{x}) & \text{in } \Omega, \end{cases}$$

and such that the following additional measurement is satisfied (the condition of final overdetermination)

$$\mathbf{u}(\mathbf{x},T) = \boldsymbol{\xi}_T(\mathbf{x}), \qquad \mathbf{x}\in\Omega.$$

- Note that this inverse problem is ill-posed
- A damping term in thermoelastic systems is also considered in [Qin, 2008, Chapter 9], [Kirane and Tatar, 2001], [Oliveira and Charão, 2008],...
- If $\mathbf{g}$ is linear, then it is possible to consider the case of non-homogeneous Dirichlet boundary conditions
- Also additional given source terms can be considered if $\mathbf{g}$ is linear

Coupled variational formulation: find $\langle \mathbf{u}, \theta, \mathbf{f} \rangle \in \mathbf{H}_0^1(\Omega) \times H_0^1(\Omega) \times \mathbf{L}^2(\Omega)$ such that $\mathbf{u}(\mathbf{x}, T) = \xi_T(\mathbf{x})$ and

$$(\partial_{tt}\mathbf{u}, \varphi) + (\mathbf{g}(\partial_t \mathbf{u}), \varphi) + \alpha (\nabla \mathbf{u}, \nabla \varphi) + \beta (\nabla \cdot \mathbf{u}, \nabla \cdot \varphi) + \gamma (\nabla \theta, \varphi) = (\mathbf{f}, \varphi),$$
$$(\partial_t \theta, \psi) + \rho (\nabla \theta, \nabla \psi) + (k \star \nabla \theta, \nabla \psi) - \gamma (\partial_t \mathbf{u}, \nabla \psi) = 0,$$

for all $\varphi \in \mathbf{H}_0^1(\Omega)$ and $\psi \in H_0^1(\Omega)$.

Theorem (Uniqueness)

Let $\langle \mathbf{u}_1, \theta_1, \mathbf{f}_1 \rangle$ and $\langle \mathbf{u}_2, \theta_2, \mathbf{f}_2 \rangle$ satisfy the thermoelastic system. Set $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $\mathbf{f} = \mathbf{f}_1 - \mathbf{f}_2$ and $\theta = \theta_1 - \theta_2$ such that $\mathbf{u}(\mathbf{x}, 0) = \mathbf{0}$, $\mathbf{u}(\mathbf{x}, T) = \mathbf{0}$, $\partial_t \mathbf{u}(\mathbf{x}, 0) = \mathbf{0}$ and $\theta(\mathbf{x}, 0) = 0$. Then $\mathbf{f} = \mathbf{0}$ a.e. in Ω and $\langle \mathbf{u}, \theta \rangle = \langle \mathbf{0}, 0 \rangle$ a.e. in $\Omega \times (0, T)$.

- Subtract, equation by equation, the variational formulation corresponding with the different solutions
- We want to add up both resulting equation such that the mixed term is cancelled out
- A good choice of the test functions is needed:

$$\varphi = \partial_t \mathbf{u} \quad \text{and} \quad \psi = \theta$$

- Integrate in time over $(0, T)$ such that

$$\int_\Omega \int_0^T \mathbf{f}(\mathbf{x}) \cdot \partial_t \mathbf{u}(\mathbf{x}, t) \mathrm{d}t = \int_\Omega [\mathbf{f}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}, T) - \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}, 0)] = 0$$

Thermoelasticity of type-I

$$\|\partial_t \mathbf{u}(T)\|^2 + \int_0^T (\mathbf{g}(\partial_t \mathbf{u_1}) - \mathbf{g}(\partial_t \mathbf{u_2}), \partial_t \mathbf{u}_1 - \partial_t \mathbf{u}_2) + \|\theta(T)\|^2 + \rho \int_0^T \|\nabla\theta\|^2 = 0$$

- $\|\partial_t \mathbf{u}(T)\| = 0$ gives no guarantee that $\mathbf{u} = \mathbf{0}$
- Assume $\mathbf{g}$ componentwise strictly monotone increasing
- Then $\mathbf{u}_t = \mathbf{0}$ a.e. in $\Omega \times (0, T)$.Therefore,

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{0} \Rightarrow \mathbf{u}(\mathbf{x}, t) = \mathbf{0} \text{ a.e. in } \Omega \times (0, T)$$

- $\theta = 0$ on $\partial\Omega \Rightarrow \theta = 0$ a.e. in $\Omega \times (0, T)$
- This implies that

$$(\mathbf{f}, \varphi) = 0, \qquad \forall \varphi \in \mathbf{H}_0^1(\Omega).$$

From this, we conclude that $\mathbf{f} = \mathbf{0}$ in $\mathbf{L}^2(\Omega)$

Thermoelasticity of type-II

$$\|\partial_t \mathbf{u}(T)\|^2 + \int_0^T \left(\mathbf{g}\left(\partial_t \mathbf{u_1}\right) - \mathbf{g}\left(\partial_t \mathbf{u_2}\right), \partial_t \mathbf{u}_1 - \partial_t \mathbf{u}_2\right) + \|\theta(T)\|^2 + \underbrace{\int_0^T (k \star \nabla\theta, \nabla\theta)}_{\geqslant 0} = 0$$

- We have $\mathbf{u} = \mathbf{0}$, no guarantee that $\theta = 0$
- Assume that $k \in C^2([0, T])$ is strongly positive definite, i.e.

$$\int_0^T \phi(t)(k \star \phi)(t)\mathrm{d}t \geqslant C_0 \int_0^T (k \star \phi)^2 (t)\mathrm{d}t, \qquad \forall T > 0, \forall \phi \in L^1_{\mathrm{loc}}(\Omega)$$

- Then

$$\int_0^T \|k \star \nabla\theta\|^2 = 0$$

$$\Rightarrow \int_0^t k(t-s)\nabla\theta(\mathbf{x}, s)\mathrm{d}s = 0 \text{ for all } t \in [0, T] \text{ and } \mathbf{x} \in \Omega$$

- Laplace transform is one-to-one $\Rightarrow \nabla\theta = 0$ in $\Omega \times (0, T)$

Thermoelasticity of type-III

$$\|\partial_t \mathbf{u}(T)\|^2 + \int_0^T \left(\mathbf{g}\left(\partial_t \mathbf{u_1}\right) - \mathbf{g}\left(\partial_t \mathbf{u_2}\right), \partial_t \mathbf{u}_1 - \partial_t \mathbf{u}_2\right) + \|\theta(T)\|^2 + \rho \int_0^T \|\nabla\theta\|^2 + \underbrace{\int_0^T (k \star \nabla\theta, \nabla\theta)}_{\geqslant 0} = 0$$

- As in the case of thermoelasticity of type-I
- It is sufficient that $k \in C^2([0, T])$ is positive definite

$$\int_0^T \phi(t)(k \star \phi)(t)\mathrm{d}t \geqslant 0 \qquad \forall T > 0, \forall \phi \in L^1_{\mathrm{loc}}(\Omega)$$

such that

$$\int_0^T (k \star \nabla\theta, \nabla\theta) \geqslant 0$$

The algorithm is based on a sequence of well-posed direct problems

Theorem (Well-posedness of the direct problem (given $\mathbf{f}$))

Assume that $\partial_t \mathbf{f} \in \mathrm{L}_2([0,T], \mathbf{L}^2(\Omega))$, $\overline{\mathbf{u}}_0 \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$, $\overline{\mathbf{u}}_1 \in \mathbf{H}^1(\Omega)$, $\overline{\theta}_0 \in \mathrm{H}^2(\Omega) \cap H_0^1(\Omega)$ *and* $\mathbf{0} < \mathbf{g}'(s) \leqslant \mathbf{C}$ *a.e. in* $\mathbb{R}$*. Then, the thermoelastic system has a unique solution* $\langle \mathbf{u}, \theta \rangle$ *such that*

$$\mathbf{u} \in C^1([0,T], \mathbf{H}_0^1(\Omega)), \quad \partial_{tt}\mathbf{u} \in C([0,T], \mathbf{L}^2(\Omega)),$$
$$\theta \in C([0,T], H_0^1(\Omega)), \qquad \theta_t \in C([0,T], L_2(\Omega)).$$

In the special situation that $\overline{\mathbf{u}}_0(\mathbf{x}) = \mathbf{0}$, $\overline{\mathbf{u}}_1(\mathbf{x}) = \mathbf{0}$ *and* $\overline{\theta}_0 = 0$*, the following energy estimate is valid*

$$\max_{t\in[0,T]} \left\{ \|\nabla \mathbf{u}(t)\|^2 + \|\nabla \partial_t \mathbf{u}(t)\|^2 + \|\nabla \theta(t)\|^2 + \|\partial_t \theta(t)\|^2 \right\} \leqslant C \|\mathbf{f}\|^2 .$$

- [Muñoz Rivera and Qin, 2002] proved the global existence and uniqueness of solutions for the one dimensional type-III thermoelastic system when $\mathbf{f} = \mathbf{0}$ and $\mathbf{g} = \mathbf{0}$
- In the same situation, a more dimensional case is studied in [Zhang and Zuazua, 2003]
- More general (linear) setting: [Lions and Magenes, 1972, Slodička, 1989a, Slodička, 1989b]

By the principle of linear superposition, we can study

$$\begin{cases} \partial_{tt}\mathbf{u} + \mathbf{g}\left(\partial_t \mathbf{u}\right) - \alpha\Delta\mathbf{u} - \beta\nabla\left(\nabla\cdot\mathbf{u}\right) + \gamma\nabla\theta &= \mathbf{f} \qquad (\mathbf{x},t)\in Q_T;\\ \partial_t\theta - \rho\Delta\theta - k\star\Delta\theta + \gamma\nabla\cdot\partial_t\mathbf{u} &= 0 \qquad (\mathbf{x},t)\in Q_T;\\ \mathbf{u}(\mathbf{x},t) &= \mathbf{0} \qquad (\mathbf{x},t)\in \Sigma_T;\\ \theta(\mathbf{x},t) &= 0 \qquad (\mathbf{x},t)\in \Sigma_T;\\ \mathbf{u}(\mathbf{x},0) = \partial_t\mathbf{u}(\mathbf{x},0) = \mathbf{0}, \quad \theta(\mathbf{x},0) &= 0 \qquad \mathbf{x}\in\Omega; \end{cases}$$

together with the transformed final measurement, i.e.

$$\mathbf{u}(\mathbf{x},T) = \widetilde{\xi}_T(\mathbf{x}), \qquad \mathbf{x}\in\Omega$$

- Define the corresponding solution operator $M(t) : \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega)$ by

$$M(t)\mathbf{f} = \mathbf{u}(\cdot,t).$$

- $M(t) \in \mathcal{L}\left(\mathbf{L}^2(\Omega), \mathbf{L}^2(\Omega)\right)$ because the initial conditions are zero
- Finding a solution to the inverse problem is then equivalent to solving the following operator equation

$$M(T)\mathbf{f} = \widetilde{\xi}_T.$$

Algorithm for finding the source term if $\mathbf{g}$ is linear

(i) Choose an initial guess $\mathbf{f}_0 \in \mathbf{L}^2(\Omega)$. Let $\langle \mathbf{u}_0, \theta_0 \rangle$ be the solution to the thermoelastic system with $\mathbf{f} = \mathbf{f}_0$

(ii) Assume that $\mathbf{f}_k$ and $\langle \mathbf{u}_k, \theta_k \rangle$ have been constructed. Let $\langle \mathbf{w}_k, \eta_k \rangle$ solve the thermoelastic system with $\mathbf{f}(\mathbf{x}) = \mathbf{u}_k(\mathbf{x}, T) - \widetilde{\boldsymbol{\xi}}_T(\mathbf{x})$

(iii) Define

$$\mathbf{f}_{k+1}(\mathbf{x}) = \mathbf{f}_k(\mathbf{x}) - \kappa \mathbf{w}_k(\mathbf{x}, T), \quad \mathbf{x} \in \Omega$$

where $\kappa > 0$ (relaxation parameter), and let $\langle \mathbf{u}_{k+1}, \theta_{k+1} \rangle$ solve the thermoelastic system with $\mathbf{f} = \mathbf{f}_{k+1}$

(iv) The procedure continues by repeating steps (ii) and (iii) until a desired level of accuracy is achieved (see further)

Problem

How to proof the convergence of this scheme?

Convergence of the proposed algorithm

Proof: The linearity of $M(T)$ implies

$$\begin{aligned} \mathbf{f}_{k+1} &= \mathbf{f}_k - \kappa \mathbf{w}_k(\cdot, T) \\ &= \mathbf{f}_k - \kappa M(T)\left(\mathbf{u}_k(\cdot, T) - \widetilde{\boldsymbol{\xi}}_T\right) \\ &= \mathbf{f}_k - \kappa M(T)\left(M(T)\mathbf{f}_k - M(T)\mathbf{f}\right) \\ &= \mathbf{f}_k - \kappa M(T)M(T)\left(\mathbf{f}_k - \mathbf{f}\right) \end{aligned}$$

Therefore,

$$\mathbf{f}_{k+1} - \mathbf{f} = \left(I - \kappa M(T)M(T)\right)\left(\mathbf{f}_k - \mathbf{f}\right)$$

- This is a Landweber-Friedmann iteration scheme for solving the operator equation $M(T)\mathbf{f} = \widetilde{\boldsymbol{\xi}}_T$
- If $0 < \kappa < \|M(T)\|^{-2}$, then the sequence $\mathbf{f}_k$ converges to $\mathbf{f}$ in $\mathbf{L}^2(\Omega)$ for arbitrary $\mathbf{f}_0 \in \mathbf{L}^2(\Omega)$ [Engl et al., 1996, Theorem 6.1]-[Slodička and Melicher, 2010, Theorem 3]
- $\mathbf{u}_k \to \mathbf{u}$ and $\theta_k \to \theta$ in $C\left([0, T], \mathbf{H}_0^1(\Omega)\right)$

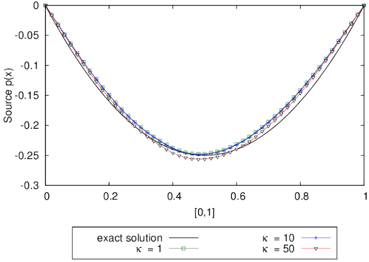
Stopping criterion

- The case is considered when there is some error in the additional measurement, i.e.
$$\|\boldsymbol{\xi}_T - \boldsymbol{\xi}_T^e\| \leqslant e,$$
with the noise level $e > 0$
- This implies that also $\widetilde{\boldsymbol{\xi}}_T$ is perturbated, denoted by $\widetilde{\boldsymbol{\xi}}_T^e$
- The solutions $\mathbf{f}_k^e, \mathbf{u}_k^e$ and θ_k^e at iteration k are obtained by using the algorithm
- The discrepancy principle [Morozov, 1966] suggests to finish the iterations at the smallest index $k = k(e, \kappa)$ for which

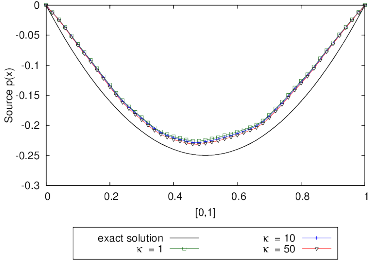
$$E_{k,\mathbf{u}_T} = \left\|\mathbf{u}_k^e(\cdot, T) - \widetilde{\boldsymbol{\xi}}_T^e\right\| \leqslant e$$

Numerical experiment: setting

- 1D linear model of type-I thermoelasticity is considered: $\Omega = [0, 1]$ and $T = 1$, $g = I$
- The forward coupled problems in this procedure are discretized in time according to the backward Euler method with timestep 0.001
- At each time-step, the resulting elliptic coupled problems are solved numerically by the finite element method (FEM) using first order (P1-FEM) Lagrange polynomials for the space discretization. A fixed uniform mesh consisting of 50 intervals is used
- The unknown source in the experiment is $f(x) = x(x-1)$
- Final in time measurement: $\xi_1(x) = 4x(x-1) +$ uncorrelated noise
- Implementation: in FEniCS



(a)



(b)

Figure : The exact solution and the numerical solution for the source for $\tilde{e} = 1\%$ (a) and $\tilde{e} = 5\%$ (b) for different values of κ.

Table : The stopping iteration number $k = k(e(\tilde{e}), \kappa)$ for the numerical experiment

$\kappa \setminus \tilde{e}$	1%	3%	5%
1	151	108	107
10	14	10	10
50	3	2	2

Conclusion:

- It is possible to recover uniquely an unknown vector source in all types of damped thermoelastic systems when an additional final in time measurement of the displacement is measured
- A numerical algorithm in a linear case gives accurate shape recovery

Future research:

- More numerical experiments
- Testing different stopping criteria (up to now, no better results)
- Recovery of time-dependent sources in thermoelastic systems
- Inverse kernel problems for thermoelasticity
- Goal: with numerical scheme!

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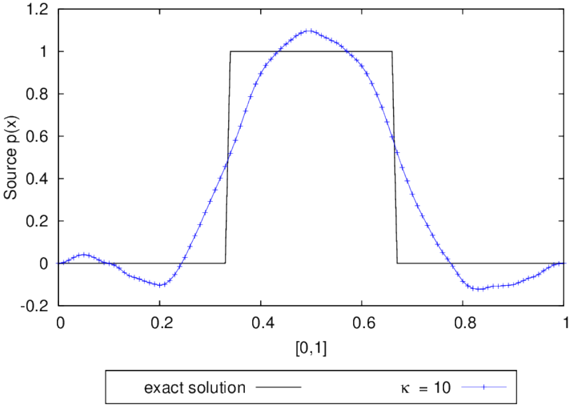

Figure : The exact solution and the numerical solution for the discontinuous source for $\tilde{e} = 1\%$ and $\kappa = 10$.