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Inverse source problems for isotropic thermoelastic systems

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Outline

- $\Omega \subset \mathbb{R}^d, d \in \{1,2,3\}$: isotropic and homogeneous thermoelastic body
- $\Gamma = \partial\Omega$: Lipschitz continuous boundary
- T: final time
- Coupled thermoelastic system [Muñoz Rivera and Qin, 2002]: specific formulas are used in the study of thermoelasticity to describe how objects change in shape (displacement vector $\mathbf{u}$) with changes in temperature θ

$$\begin{cases} \partial_{tt}\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla(\nabla\cdot\mathbf{u}) + \gamma\nabla\theta &= \mathbf{f} \qquad \text{in } \Omega\times(0,T) \\ \partial_t\theta - \rho\Delta\theta - k*\Delta\theta + \gamma\nabla\cdot\partial_t\mathbf{u} &= h \qquad \text{in } \Omega\times(0,T) \end{cases}$$

- $\mathbf{f}$: load (body force) vector; h: heat source
- The Lamé parameters α and β, the coupling (absorbing) coefficient γ and the thermal coefficient ρ are assumed to be positive constants
- The sign '$*$' denotes the convolution product

$$(k*\theta)(\mathbf{x},t) := \int_0^t k(t-s)\theta(\mathbf{x},s)\mathrm{d}s, \qquad (\mathbf{x},t)\in\Omega\times(0,T)$$

Types of thermoelasticity

$$\begin{cases} \partial_{tt}\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla(\nabla\cdot\mathbf{u}) + \gamma\nabla\theta = \mathbf{f} & \text{in } \Omega\times(0,T); \\ \partial_t\theta - \rho\Delta\theta - k*\Delta\theta + \gamma\nabla\cdot\partial_t\mathbf{u} = h & \text{in } \Omega\times(0,T); \\ \mathbf{u}(\mathbf{x},0) = \overline{\mathbf{u}}_0(\mathbf{x}), \quad \partial_t\mathbf{u}(\mathbf{x},0) = \overline{\mathbf{u}}_1(\mathbf{x}), \quad \theta(\mathbf{x},0) = \overline{\theta}_0(\mathbf{x}) & \text{in } \Omega \end{cases}$$

Three types of thermoelasticity:

- type-I: $k = 0$ and $\rho \neq 0$:

$$\partial_t\theta - \rho\Delta\theta + \gamma\nabla\cdot\partial_t\mathbf{u} = h$$

- type-II: $k \neq 0$ and $\rho = 0$:

$$\partial_t\theta - k*\Delta\theta + \gamma\nabla\cdot\partial_t\mathbf{u} = h$$

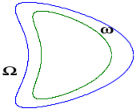
- type-III: $k \neq 0$ and $\rho \neq 0$:

$$\partial_t\theta - \rho\Delta\theta - k*\Delta\theta + \gamma\nabla\cdot\partial_t\mathbf{u} = h$$

Inverse source problems for isotropic thermoelasticity are studied

- [Bellassoued and Yamamoto, 2011] investigated an inverse heat source problem for type-I thermoelasticity: they determine $h(\mathbf{x})$ by measuring

$$\mathbf{u}_{|\omega\times(0,T)} \text{ and } \theta(\cdot, t_0),$$



where ω is a subdomain of Ω such that $\Gamma \subset \partial\omega$ and $t_0 \in (0, T)$

- [Wu and Liu, 2012] studied an inverse source problem of determining $\mathbf{f}(\mathbf{x})$ for type-II thermoelasticity from a displacement measurement

$$\mathbf{u}_{|\omega\times(0,T)}$$

- Using a Carleman estimate, a Hölder stability for the inverse source problem is proved in both contributions, which implies the uniqueness of a solution to the inverse source problem
- No numerical scheme is provided to recover the unknown source

Problem (A)

Can we find a unique $\mathbf{f}(\mathbf{x})$ *and/or* $h(\mathbf{x})$ *from the additional final in time measurements*

$$\mathbf{u}(\mathbf{x}, T) = \boldsymbol{\xi}_T(\mathbf{x}) \text{ and/or } \theta(\mathbf{x}, T) = \zeta_T(\mathbf{x})$$

for all types of thermoelasticity and can we provide a numerical scheme?

Problem (B)

Can we find a unique $\mathbf{f}(t)$ *and/or* $h(t)$ *from the additional global measurement in integral form*

$$\int_\Omega \mathbf{u}(\mathbf{x}, t) \, \mathrm{d}\mathbf{x} = \mathbf{m}(t) \text{ and/or } \int_\Omega \theta(\mathbf{x}, t) \, \mathrm{d}\mathbf{x} = m(t)$$

for all types of thermoelasticity and can we provide a numerical scheme?

Goal: The way of retrieving the unknown source is not by the minimization of a certain cost functional (which is typical for IPs), but by using an alternative technique

Solution (Problem (A))

Up to now, using our approach, it is possible to recover $\mathbf{f}(\mathbf{x})$ *uniquely for all types of thermoelasticity from the additional final in time measurement (the condition of final overdetermination)*

$$\mathbf{u}(\mathbf{x}, T) = \xi_T(\mathbf{x}),$$

in the presence of a damping term $\mathbf{g}(\partial_t \mathbf{u})$ *in the hyperbolic equation of the thermoelastic system, i.e.*

$$\begin{cases} \partial_{tt}\mathbf{u} + \mathbf{g}(\partial_t \mathbf{u}) - \alpha\Delta\mathbf{u} - \beta\nabla(\nabla\cdot\mathbf{u}) + \gamma\nabla\theta = \mathbf{f}(\mathbf{x}) & \text{in } \Omega\times(0,T); \\ \partial_t\theta - \rho\Delta\theta - k*\Delta\theta + \gamma\nabla\cdot\partial_t\mathbf{u} = 0 & \text{in } \Omega\times(0,T); \\ \mathbf{u}(\mathbf{x},t) = \mathbf{0} & \text{on } \Gamma\times(0,T); \\ \theta(\mathbf{x},t) = 0 & \text{on } \Gamma\times(0,T); \\ \mathbf{u}(\mathbf{x},0) = \partial_t\mathbf{u}(\mathbf{x},0) = \mathbf{0}, \quad \theta(\mathbf{x},0) = 0 & \text{in } \Omega, \end{cases}$$

See: Van Bockstal, K. and Slodička, M. *Recovery of a space-dependent vector source in thermoelastic systems.*
Inverse Problems in Science and Engineering, 2015, 23, 956–968

- Note that this inverse problem is ill-posed
- A damping term in thermoelastic systems is also considered in [Qin, 2008, Chapter 9], [Kirane and Tatar, 2001], [Oliveira and Charão, 2008],...
- If **g** is linear, then it is possible to consider the case of non-homogeneous Dirichlet boundary conditions and intial conditions
- Also additional given source terms can be considered if **g** is linear
- A variational approach is used, which implies uniqueness for all types of thermoelasticity if $\mathbf{g} : \mathbb{R}^d \mapsto \mathbb{R}^d$ is strictly monotone increasing and k is strongly positive definite
- A stable iterative algorithm is proposed to recover the unknown vector source **f** by extending the iterative procedure of [Johansson and Lesnic, 2007] for the heat equation to thermoelastic systems, but without using an adjoint problem

Algorithm for finding the source term if $\mathbf{g}$ is linear

(*i*) Choose an initial guess $\mathbf{f}_0 \in \mathbf{L}^2(\Omega)$. Let $\langle \mathbf{u}_0, \theta_0 \rangle$ be the solution to the thermoelastic system with $\mathbf{f} = \mathbf{f}_0$

(*ii*) Assume that $\mathbf{f}_k$ and $\langle \mathbf{u}_k, \theta_k \rangle$ have been constructed. Let $\langle \mathbf{w}_k, \eta_k \rangle$ solve the thermoelastic system with $\mathbf{f}(\mathbf{x}) = \mathbf{u}_k(\mathbf{x}, T) - \boldsymbol{\xi}_T(\mathbf{x})$

(*iii*) Define

$$\mathbf{f}_{k+1}(\mathbf{x}) = \mathbf{f}_k(\mathbf{x}) - \kappa \mathbf{w}_k(\mathbf{x}, T), \quad \mathbf{x} \in \Omega$$

where $\kappa > 0$ (relaxation parameter), and let $\langle \mathbf{u}_{k+1}, \theta_{k+1} \rangle$ solve the thermoelastic system with $\mathbf{f} = \mathbf{f}_{k+1}$

(*iv*) The procedure continues by repeating steps (*ii*) and (*iii*) until a desired level of accuracy is achieved

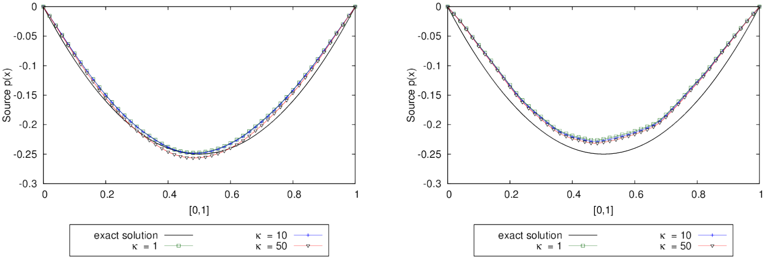
This algorithm is based on an iterative regularization method, namely the Landweber-Fridman iteration [Landweber, 1951, Fridman, 1956].

Theorem ([Van Bockstal and Slodička, 2015, Theorem 3.3])

If κ is sufficiently small, then the sequence $\mathbf{f}_k$ converges to $\mathbf{f}$ in $\mathbf{L}^2(\Omega)$ for arbitrary $\mathbf{f}_0 \in \mathbf{L}^2(\Omega)$.

Numerical experiment: setting

- 1D linear model of type-I thermoelasticity is considered: $\Omega = [0,1]$, $T = 1$, $g = I$, and $\alpha + \beta = \gamma = \rho = 1$
- The forward coupled problems in this procedure are discretized in time according to the backward Euler method with timestep 0.001
- At each time-step, the resulting elliptic coupled problems are solved numerically by the finite element method (FEM) using first order (P1-FEM) Lagrange polynomials for the space discretization. A fixed uniform mesh consisting of 50 intervals is used
- The unknown source in the experiment is $f(x) = x(x-1)$
- Final in time measurement: $\xi_1(x) = 4x(x-1)$ + uncorrelated noise
- The discrepancy principle of Morozov [Morozov, 1966] is used to stop the algorithm
- FEniCS project — The finite element library DOLFIN [Logg and Wells, 2010, Logg et al., 2012b] from the FEniCS project [Logg et al., 2012a] is used



(a) (b)

Figure : The exact solution and the numerical solution for the source for $\tilde{e} = 1\%$ (a) and $\tilde{e} = 5\%$ (b) for different values of κ.

Table : The stopping iteration number k for the numerical experiment

$\kappa \setminus \tilde{e}$	1%	3%	5%
1	151	108	107
10	14	10	10
50	3	2	2

Conclusion:

- It is possible to recover uniquely an unknown vector source in all types of damped thermoelastic systems when an additional final in time measurement of the displacement is measured
- A numerical algorithm in a linear case gives accurate shape recovery

Future research:

- Extending the results to anisotropic thermoelastic systems and performing physically relevant numerical experiments (KVB and L. Marin)
- More general boundary conditions
- Testing different stopping criteria (up to now, no better results)
- Recovery of a space dependent heat source (presented technique gives no uniqueness of a solution)

Solution (Problem (B))

Up to now, using our approach, it is possible to recover $h(t)$ uniquely in a one-dimensional thermoelastic system of type-I or type-III from the averaged temperature

$$\int_0^L \theta(\mathbf{x}, t) = m(t),$$

i.e. such that

$$\begin{cases} \ddot{u} - \alpha u'' + \gamma\theta' & = r & \text{in } (0, L] \times (0, T] \\ \dot{\theta} - \rho\theta'' - k * \theta'' + \gamma\dot{u}' & = h(t)f(x) + s & \text{in } (0, L] \times (0, T] \\ u'(0, t) = u'(L, t) = \theta(0, t) = \theta(L, t) & = 0 & \text{in } (0, T] \\ u(x, 0) = u_0(x),\ \dot{u}(x, 0) = v_0(x),\ \theta(x, 0) & = \theta_0(x) & \text{in } (0, L). \end{cases}$$

See: Van Bockstal, K. and Slodička, M. *Recovery of a time-dependent heat source in one-dimensional thermoelasticity of type-III.* Inverse Problems in Science and Engineering, 2016 (accepted for publication)

Main idea: The inverse problem is recasted into a direct problem and the well-posedness of the problem is shown by using Rothe's method

We assume that

- $k \in \mathrm{C}([0,T])$
- $m \in \mathrm{C}^1([0,T])$
- $s \in \mathrm{L}^2\left((0,T), \mathrm{L}^2(\Omega)\right)$
- $r \in \mathrm{L}^2\left((0,T), \mathrm{H}^1(\Omega)\right)$
- $f \in \mathrm{H}^1(\Omega)$ with $\int_0^L f \neq 0$
- $\theta_0 \in \mathrm{H}^2(\Omega)$, $u_0 \in \mathrm{H}^1(\Omega)$ and $v_0 \in \mathrm{H}^1(\Omega)$

and refer to these conditions as $(\star)$

Reformulation into a direct problem

- First, the parabolic equation is integrated over Ω. This gives an expression for the unknown function h in terms of the unknowns u and θ, i.e.

$$h(t) = \frac{\dot{m}(t) - \rho \int_0^L \theta''(t) - \left(k * \int_0^L \theta''\right)(t) + \gamma \int_0^L \dot{u}'(t) - \int_0^L s(t)}{\int_0^L f} \in \mathbb{R},$$

  if $\int_0^L f \neq 0$

- Next, using Green's formulas, the following coupled variational formulation is obtained: find $\langle u(t), \theta(t)\rangle \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1_0(\Omega)$ such that

$$(\ddot{u}(t), \phi) + \alpha \left(u'(t), \phi'\right) + \gamma \left(\theta'(t), \phi\right) = (r(t), \phi)$$

  and

$$\left(\dot{\theta}(t), \psi\right) + \rho \left(\theta'(t), \psi'\right) + \left(k * \theta'(t), \psi'\right) - \gamma \left(\dot{u}(t), \psi'\right)$$
$$= \frac{\dot{m}(t) - \rho \int_0^L \theta''(t) - \left(k * \int_0^L \theta''\right)(t) + \gamma \int_0^L \dot{u}'(t) - \int_0^L s(t)}{\int_0^L f} (f, \psi) + (s(t), \psi)$$

  for almost all $t \in [0, T]$ and for all $\phi \in \mathrm{H}^1(\Omega)$ and $\psi \in \mathrm{H}^1_0(\Omega)$.

Well-posedness of the problem

- Rothe's method [Kačur, 1985]: divide $[0, T]$ into $n \in \mathbb{N}$ equidistant subintervals $(t_{i-1}, t_i]$ for $t_i = i\tau$, where $\tau = T/n < 1$ and for any function z

$$z_i \approx z(t_i), \qquad \partial_t z(t_i) \approx \delta z_i := \frac{z_i - z_{i-1}}{\tau}$$

- The following linear recurrent scheme is proposed to approximate the original problem at time t_i, $1 \leqslant i \leqslant n$, is proposed ($\phi \in \mathrm{H}^1(\Omega)$ and $\psi \in \mathrm{H}^1_0(\Omega)$):

$$\left(\delta^2 u_i, \phi\right) + \alpha \left(u_i', \phi'\right) + \gamma \left(\theta_i', \phi\right) = (r_i, \phi)\,,$$

$$(\delta\theta_i, \psi) + \rho \left(\theta_i', \psi'\right) + \left(\sum_{l=1}^{i} k_l \theta_{i-l}' \tau, \psi'\right) - \gamma \left(\delta u_i, \psi'\right) = h_{i-1} (f, \psi) + (s_i, \psi)\,,$$

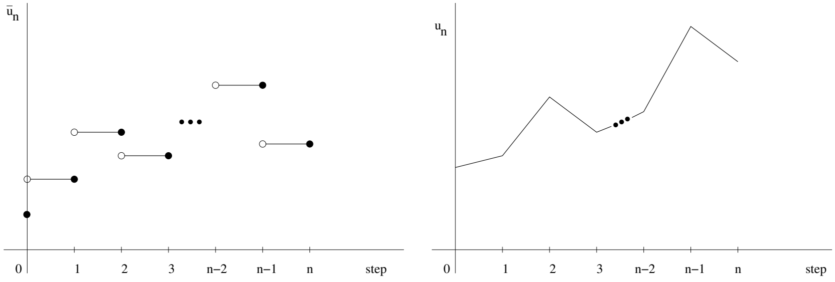
$$u_0 = u_0, \quad \delta u_0 = v_0, \quad \theta_0 = \theta_0,$$

with

$$h_{i-1} := \left(m'(t_i) - \rho \int_0^L \theta_{i-1}'' - \sum_{l=1}^{i} \tau k_l \int_0^L \theta_{i-l}'' + \gamma \int_0^L \delta u_{i-1}' - \int_0^L s_i\right) / \left(\int_0^L f\right)$$

Rothe functions

- Piecewise constant and linear in time spline of the solutions $u_i, i = 1, \ldots, n$



(a) (b)

Figure : Rothe's piecewise constant function $\overline{u}_n$ (a) and Rothe's piecewise linear in time function u_n (b).

- Similarly, the functions v_n, $\overline{v}_n$, θ_n, $\overline{\theta}_n$, $\overline{k}_n$, $\overline{r}_n$, $\overline{h}_n$, $\overline{m}_n$ and $\overline{\dot{m}}_n$ are defined

The discrete variational formulation can be rewritten on the whole time frame as

$$(\dot{v}_n(t), \phi) + \alpha \left(\overline{u}_n'(t), \phi'\right) + \gamma \left(\overline{\theta}_n'(t), \phi\right) = (\overline{r}_n(t), \phi)\,,$$

and

$$\left(\dot{\theta}_n(t), \psi\right) + \rho \left(\overline{\theta}_n'(t), \psi'\right) + \left(\sum_{l=1}^{\lceil t \rceil_\tau} \overline{k}_n(t_l) \overline{\theta}_n'(t - t_l)\tau, \psi'\right) - \gamma \left(\dot{u}_n(t), \psi'\right) = \overline{h}_n(t-\tau)\,(f, \psi) + (\overline{s}_n(t), \psi)\,,$$

with

$$\left(\int_0^L f\right) \overline{h}_n(t-\tau) = \overline{\dot{m}}_n(t) - \rho \int_0^L \overline{\theta}_n''(t-\tau) - \sum_{l=1}^{\lceil t \rceil_\tau} \tau \overline{k}_n(t_l) \int_0^L \overline{\theta}_n''(t - t_l) + \gamma \int_0^L \dot{u}_n'(t-\tau) - \int_0^L \overline{s}_n(t),$$

with $\lceil \cdot \rceil_\tau$ defined by $\lceil t \rceil_\tau = i$ when $t \in (t_{i-1}, t_i]$

We pass to the limit $n \to \infty$ (term by term)

Theorem (Existence)

Let (⋆) be satisfied. Then there exists a triplet

$$\langle u, \theta, h\rangle \in \left[\mathrm{C}\left([0,T], \mathrm{C}(\overline{\Omega})\right) \cap \mathrm{L}^\infty\left((0,T), \mathrm{C}^1(\overline{\Omega})\right)\right] \times \left[\mathrm{C}\left([0,T], \mathrm{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0,T), \mathrm{C}^1(\overline{\Omega})\right)\right] \times \mathrm{L}^2(0,T)$$

with

$$\dot{u} \in \mathrm{L}^2\left((0,T), \mathrm{H}^1(\Omega)\right), \ddot{u} \in \mathrm{L}^2\left((0,T), \mathrm{L}^2(\Omega)\right), \dot{\theta} \in \mathrm{L}^2\left((0,T), \mathrm{L}^2(\Omega)\right)$$

such that $\langle u, \theta, h\rangle$ *is a weak solution to the problem.*

Theorem (Uniqueness)

Let (⋆) be satisfied. Then there exists at most one triple

$$\langle u, \theta, h\rangle \in \mathrm{C}\left([0,T], \mathrm{H}^2(\Omega)\right) \times \left[\mathrm{C}\left([0,T], \mathrm{H}^1_0(\Omega)\right) \cap \mathrm{L}^2\left((0,T), V_\theta\right)\right] \times \mathrm{L}^2(0,T),$$

with

$$\dot{u} \in \mathrm{L}^2\left((0,T), \mathrm{H}^2(\Omega)\right), \ddot{u} \in \mathrm{L}^2\left((0,T), \mathrm{H}^1(\Omega)\right), \dot{\theta} \in \mathrm{L}^2\left((0,T), \mathrm{H}^1_0(\Omega)\right)$$

solving the problem.

- Uniqueness in smaller function spaces!
- Results only valid for type-I and type-III thermoelasticity

Numerical experiment: setting

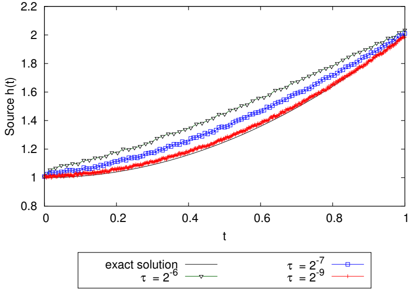
- 1D linear model of type-I thermoelasticity is considered: $\Omega = [0,1]$, $T = 1$, and $\alpha = \gamma = \rho = 1$
- At each time-step, the resulting elliptic problems are solved numerically by the finite element method (FEM) using first order (P1-FEM) Lagrange polynomials for the space discretization for which a fixed uniform mesh of 2000 subintervals is used
- The displacement and temperature are the same in each experiment:

  $u(x,t) = \left(t^2 + t + 1\right)\left(1 + \cos\left(\pi x\right)\right)$ and $\theta(x,t) = 1 + 2\left(t^2 + 1\right) x \left(1 - x\right).$

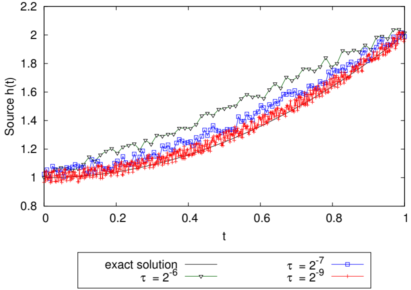
  The additional measurement is given by

  $$m(t) = \frac{4}{3} + \frac{1}{3} t^2$$

- An uncorrelated noise is added to $m'(t) = \frac{2}{3} t$ in order to simulate the errors present in real measurements. This noise is generated randomly with given magnitude of 1% and 5%

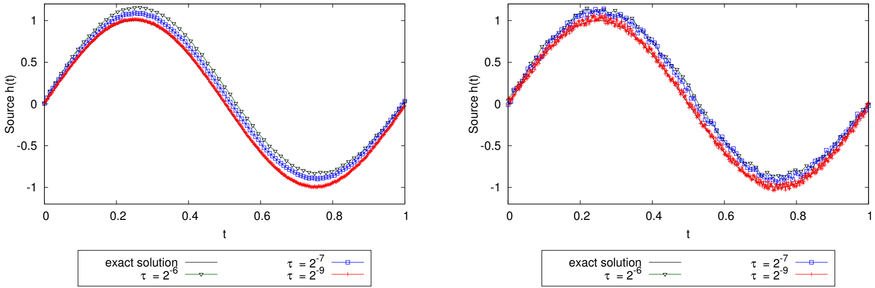


(a) exact solution $h(t) = 1 + t^2$ and its numerical approximations (noise with magnitude 1%)



(b) exact solution $h(t) = 1 + t^2$ and its numerical approximations (noise with magnitude 5%)

Figure : Heat source reconstruction in Experiment 1.



(a) exact solution $h(t) = \sin(2\pi t)$ and its numerical approximations (noise with magnitude 1%)

(b) exact solution $h(t) = \sin(2\pi t)$ and its numerical approximations (noise with magnitude 5%)

Figure : Heat source reconstruction in Experiment 2.

Conclusion:

- It is possible to recover uniquely an unknown heat source in type-I and III thermoelastic systems when the averaged temperature is measured
- The numerical algorithm gives accurate recovery of the unknown source if the timestep τ is sufficiently small

Future research:

- Derivation of error estimates
- The results are valid for type-I and type-III thermoelasticity if $k \in C([0, T])$. What if type-II thermoelasticity is considered?
- The results stay valid when the additional condition is replaced by $\int_0^L \chi(x)\theta(x,t)\,dx = m(t)$, where χ is a known weighted function (measurement localization), if $\chi \in C_0^1([0, L])$. What if a point measurement $\theta(x_0, t) = m(t)$ with $x_0 \in (0, L)$ is considered?
- More dimensional case, anisotropic materials, recovery of time-dependent load source
- Implementing relevant problems with correct physical parameters

References I

Bellassoued, M. and Yamamoto, M. (2011).
Carleman estimates and an inverse heat source problem for the thermoelasticity system.
Inverse Problems, 27(1):015006.

Fridman, V. M. (1956).
Method of successive approximations for a fredholm integral equation of the 1st kind.
Uspekhi Mat. Nauk, 11(1(67)):233–234.

Johansson, T. and Lesnic, D. (2007).
Determination of a spacewise dependent heat source.
J. Comput. Appl. Math., 209(1):66–80.

Kačur, J. (1985).
Method of Rothe in evolution equations, volume 80 of *Teubner Texte zur Mathematik*.
Teubner, Leipzig.

Kirane, M. and Tatar, N.-E. (2001).
A nonexistence result to a cauchy problem in nonlinear one dimensional thermoelasticity.
Journal of Mathematical Analysis and Applications, 254(1):71 – 86.

Landweber, L. (1951).
An iteration formula for fredholm integral equations of the first kind.
American Journal of Mathematics, 73(3):615–624.

References II

Logg, A., Mardal, K.-A., Wells, G. N., et al. (2012a).
Automated Solution of Differential Equations by the Finite Element Method.
Springer.

Logg, A., Wells, G., and Hake, J. (2012b).
DOLFIN: a C++/Python Finite Element Library, chapter 10.
Springer.

Logg, A. and Wells, G. N. (2010).
DOLFIN: Automated Finite Element Computing.
ACM Trans. Math. Software, 37(2):28.

Morozov, V. A. (1966).
On the solution of functional equations by the method of regularization.
Soviet Math. Dokl., 7:414–417.

Muñoz Rivera, J. E. and Qin, Y. (2002).
Global existence and exponential stability in one-dimensional nonlinear thermoelasticity with thermal memory.
Nonlinear Anal.: Theory, Methods & Applications, 51(1):11–32.

References III

Oliveira, J. C. and Charão, R. C. (2008).
Stabilization of a locally damped thermoelastic system.
Comput. Appl. Math., 27(3):319–357.

Qin, Y. (2008).
Nonlinear parabolic-hyperbolic coupled systems and their attractors.
Basel: Birkhäuser.

Van Bockstal, K. and Slodička, M. (2015).
Recovery of a space-dependent vector source in thermoelastic systems.
Inverse Problems Sci. Eng., 23(6):956–968.

Wu, B. and Liu, J. (2012).
Determination of an unknown source for a thermoelastic system with a memory effect.
Inverse Problems, 28(9):095012.