GHENT UNIVERSITY

FACULTY OF ENGINEERING AND ARCHITECTURE

Identification of an unknown spatial load distribution in a vibrating beam or plate from the final state

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Outline

- $\Omega \subset \mathbb{R}^d$: thin beam ($d = 1$) or thin plate ($d = 2$) with Lipschitz continuous boundary Γ
- T: final time
- $Q_T := \Omega \times (0, T)$ and $\Sigma_T := \Gamma \times (0, T)$
- Let X be a Banach space with norm $\|\cdot\|_X$
  - $\mathrm{C}^k([0, T], X)$ with $k \in \mathbb{N}$: consists of k-times continuously differentiable functions $w : [0, T] \to X$ with $\sum_{j=0}^{k} \max_{t\in[0,T]} \|w^{(j)}(t)\|_X < \infty$
  - $\mathrm{L}^p((0, T), X)$ with $1 \leqslant p < +\infty$: consists of functions $w : (0, T) \to X$ satisfying $\left(\int_0^T \|w(t)\|_X^p \, dt\right)^{\frac{1}{p}} < \infty$
  - $\mathrm{L}^\infty((0, T), X)$: consists of functions $w : (0, T) \to X$ that are essentially bounded, i.e.
    $\|w\|_{\mathrm{L}^\infty((0,T),X)} = \inf \left\{ B : \|w(t)\|_X \leq B \text{ for almost all } t \in (0, T) \right\} < \infty$
  - $\mathrm{H}^1((0, T), X)$: consists of functions $u : (0, T) \to X$ such that
    $\|u\|_{\mathrm{H}^1((0,T),X)} = \left(\int_0^T \|u(t)\|_X^2 + \|u'(t)\|_X^2 \, \mathrm{d}t\right)^{1/2} < \infty$

- Dynamic vibration of a simply supported non-homogeneous Euler-Bernoulli beam ($d = 1$) and Kirchhoff-Love plate ($d = 2$) is governed by the following problem (for small deflection u):

$$\begin{cases} \rho\partial_{tt}u + \mu\partial_t u + \Delta(k\Delta u) - T^r\Delta u &= p \qquad \text{in } Q_T, \\ u &= 0 \qquad \text{on } \Sigma_T, \\ k\Delta u &= 0 \qquad \text{on } \Sigma_T, \\ u(\mathbf{x},0) &= \tilde{u}_0(\mathbf{x}) \quad \mathbf{x}\in\Omega, \\ \partial_t u(\mathbf{x},0) &= \tilde{v}_0(\mathbf{x}) \quad \mathbf{x}\in\Omega \end{cases} \tag{1}$$

- $u(\mathbf{x}, t)$: the displacement in z-direction from the equilibrium position $u \equiv 0$
- $\tilde{u}_0$: initial deflection, $\tilde{v}_0$: initial velocity
- $p(\mathbf{x}, t)$: load distribution, $\rho(\mathbf{x}, t)$: mass density, $\mu(\mathbf{x}, t)$: damping coefficient, $T_r(\mathbf{x}, t)$: traction force
- $$k(\mathbf{x}, t) = \begin{cases} \text{flexural rigidity} & d = 1 \\ \text{bending stiffness} & d = 2 \end{cases}$$

Inverse Source Problem (ISP)

Determine $f(\mathbf{x})$ in

$$\begin{cases} \rho\partial_{tt}u + \mu\partial_t u + \Delta(k\Delta u) - T^r\Delta u & = f(\mathbf{x})h(t) & \text{in } Q_T, \\ u & = 0 & \text{on } \Sigma_T, \\ k\Delta u & = 0 & \text{on } \Sigma_T, \\ u(\mathbf{x},0) & = \tilde{u}_0(\mathbf{x}) & \mathbf{x}\in\Omega, \\ \partial_t u(\mathbf{x},0) & = \tilde{v}_0(\mathbf{x}) & \mathbf{x}\in\Omega, \end{cases} \tag{2}$$

from the deflection u at final time $t = T$:

$$u(\cdot, T) = \xi_T(\cdot) \tag{3}$$

Literature overview

- [Hasanov, 2009]:
  - Determine $p(x,t)$ in a vibrating cantilevered beam of the form
  $$\rho(x)\ddot{u} + (k(x)u'')'' = p(x,t)$$
  from the measured data $u(x,T)$ or $u_t(x,T)$ by minimization of a cost functional
  - The Fréchet gradients of the cost functionals are derived via the solutions of corresponding adjoint (backward beam) problems
  - [Hasanov, 2009, Lemma 7.2]: uniqueness of the solution can be obtained when a positivity condition holds on the solution
- [Hasanov and Baysal, 2015]: the theory developed in [Hasanov, 2009] is illustrated by numerical examples for the problem of determing $f(x)$ from the final state when $p(x,t) = f(x)h(t)$

- $V := \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$
- The norms $\sum_{|\boldsymbol{\alpha}|\leqslant 2} \|D^{\boldsymbol{\alpha}} u\|^2$ and $\|\Delta u\|^2$ are equivalent in V
- Additional term λu is considered in (2) because these term appears in the forward problems that are involved when solving the ISP later
- Variational formulation of problem (1):

Find
$u(t) \in V$ with $\partial_t u(t) \in V$ and $\partial_{tt} u(t) \in \mathrm{L}^2(\Omega)$
such that

$$(\rho(t)\partial_{tt} u(t), \varphi) + (\mu(t)\partial_t u(t), \varphi) + (\lambda(t) u(t), \varphi) \\ + (k(t)\Delta u(t), \Delta\varphi) - (T^r(t)\Delta u(t), \varphi) = (p(t), \varphi) \quad (4)$$

for all $\varphi \in V$ and a.a. $t \in (0, T]$

Assumptions on data ($(\mathbf{x}, t) \in \Omega \times [0, T]$)

$$
\begin{aligned}
&0 < \tilde{\rho}_0 \leqslant \rho(\mathbf{x}, t) \leqslant \tilde{\rho}_1, \qquad |\partial_t \rho(\mathbf{x}, t)| \leqslant \tilde{\rho}_2, \\
&|\mu(\mathbf{x}, t)| \leqslant \tilde{\mu}_1, \qquad |\partial_t \mu(\mathbf{x}, t)| \leqslant \tilde{\mu}_2, \\
&|\lambda(\mathbf{x}, t)| \leqslant \tilde{\lambda}_1, \qquad |\partial_t \lambda(\mathbf{x}, t)| \leqslant \tilde{\lambda}_2, \\
&0 < \tilde{k}_0 \leqslant k(\mathbf{x}, t) \leqslant \tilde{k}_1, \qquad |\partial_t k(\mathbf{x}, t)| \leqslant \tilde{k}_2, \qquad |\partial_{tt} k(\mathbf{x}, t)| \leqslant \tilde{k}_3, \\
&|\nabla k(\mathbf{x}, 0)|_{\mathrm{e}} \leqslant \tilde{k}_4, \qquad |\Delta k(\mathbf{x}, 0)| \leqslant \tilde{k}_5, \\
&|T^r(\mathbf{x}, t)| \leqslant \tilde{T}_1, \qquad |\partial_t T^r(\mathbf{x}, t)| \leqslant \tilde{T}_2, \\
&p \in \mathrm{H}^1\left((0, T), \mathrm{L}^2(\Omega)\right), \\
&\tilde{u}_0 \in \mathrm{H}^4(\Omega) \cap \mathrm{H}^1_0(\Omega), \qquad k(\mathbf{x}, 0)\Delta \tilde{u}_0(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma, \\
&\tilde{v}_0 \in \mathrm{H}^2(\Omega)
\end{aligned}
$$

and refer to these conditions as $(\star)$

Theorem (Well-posedness of the direct problem)

Let the conditions ($\star$) be fulfilled. Then, there exists a unique weak solution to problem (4) *satisfying*

$$u \in \mathrm{C}\left([0,T],V\right)$$

with

$$\partial_t u \in \mathrm{C}\left([0,T],\mathrm{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0,T),V\right)$$

and

$$\partial_{tt} u \in \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right).$$

First step: divide the governing partial differential equation (PDE) in (2) by the known (given) function $h(t)$

- Assumed that $h \neq 0$, i.e. $h(t) > 0$ (or $h(t) < 0$) for all $t \in [0, T]$
- Let

$$v(\mathbf{x}, t) = \frac{u(\mathbf{x}, t)}{h(t)} \quad \text{and} \quad \alpha(t) = \frac{h'(t)}{h(t)}$$

- PDE in (2) can be rewritten in terms of the unknown v as follows

$$\rho\partial_{tt}v + (\mu + 2\rho\alpha)\,\partial_t v + \left(\mu\alpha + \rho\alpha^2 + \rho\alpha'\right) v + \Delta(k\Delta v) - T_r\Delta v = f(\mathbf{x}), \quad (5)$$

with

$$\begin{cases} v &= 0 & \text{on } \Sigma_T, \\ k\Delta v &= 0 & \text{on } \Sigma_T, \\ v(\mathbf{x}, 0) &= \frac{\tilde{u}_0(\mathbf{x})}{h(0)} & \mathbf{x} \in \Omega, \\ \partial_t v(\mathbf{x}, 0) &= \frac{\tilde{v}_0(\mathbf{x})}{h(0)} - \frac{\tilde{u}_0(\mathbf{x})}{h(0)}\alpha(0) & \mathbf{x} \in \Omega, \\ v(\mathbf{x}, T) &= \frac{\xi_T(\mathbf{x})}{h(T)} & \mathbf{x} \in \Omega \end{cases} \quad (6)$$

- Variational formulation corresponding forward problem:

$$\text{find} \quad v(t) \in V \text{ with } \partial_t v(t) \in V \text{ and } \partial_{tt} v(t) \in \mathrm{L}^2(\Omega) \quad \text{such that}$$

$$\begin{aligned}(\rho(t)\partial_{tt} v(t), \varphi) &+ ((\mu(t) + 2\rho(t)\alpha(t))\,\partial_t v(t), \varphi) \\ &+ \left(\left(\mu(t)\alpha(t) + \rho(t)\alpha(t)^2 + \rho(t)\alpha'(t)\right) v(t), \varphi\right) \\ &+ (k(t)\Delta v(t), \Delta\varphi) - (T_r(t)\Delta v(t), \varphi) = (f, \varphi)\,, \end{aligned} \tag{7}$$

$$\text{for all } \varphi \in V \text{ and a.a. } t \in (0, T]$$

- Following Theorem 1, this formulation is well-posed for given $f \in \mathrm{L}^2(\Omega)$ if the conditions $(\star)$ are satisfied and

$$|\alpha(t)| \leqslant \alpha_1, \quad |\alpha'(t)| \leqslant \alpha_2, \quad |\alpha''(t)| \leqslant \alpha_3, \quad t \in [0, T],$$

and refer to these conditions as $(\star\star)$

Theorem (Uniqueness)

Let the conditions $(\star)$ *and* $(\star\star)$ *be satisfied. Moreover, assume that* T_r *is solely time dependent with*

$$T_r'(t) \leqslant 0, \quad t \in [0, T],$$

and

$$\xi_T \in \mathrm{L}^2(\Omega), \quad \alpha(t) \geqslant 0, \quad \partial_t \rho \leqslant 0, \quad \mu \geqslant \mu_0 > 0,$$
$$\partial_t k \leqslant 0, \quad \text{and} \quad \partial_t\left(\mu\alpha + \rho\alpha^2 + \rho\alpha'\right) \leqslant 0.$$

Then, there exists at most one $f \in \mathrm{L}^2(\Omega)$ *such that problem* (2) *together with condition* (3) *holds.*

Proof uniqueness ISP I

- Variational approach, proof by contradiction
- Suppose two solutions $\langle u_1, f_1\rangle$ and $\langle u_2, f_2\rangle$ to (2)-(3)
- Set $u = u_1 - u_2$, $v = v_1 - v_2$ and $f = f_1 - f_2$
- Then $u(\mathbf{x}, 0) = 0$, $\partial_t u(\mathbf{x}, 0) = 0$ and $u(\mathbf{x}, T) = 0$
- Therefore, also $v(\mathbf{x}, 0) = 0$, $\partial_t v(\mathbf{x}, 0) = 0$ and $v(\mathbf{x}, T) = 0$
- First, we prove that $v = 0$ (thus $u = 0$) and then we show that $f = 0$
- Subtract the variational formulation corresponding with the different solutions

Proof uniqueness ISP II

- Choose $\varphi = \partial_t v(t)$ as testfunction and integrate in time over $(0, T)$ to obtain that

$$\int_0^T (\rho(t)\partial_{tt} v(t), \partial_t v(t))\,\mathrm{d}t + \int_0^T ((\mu(t) + 2\rho(t)\alpha(t))\,\partial_t v(t), \partial_t v(t))\,\mathrm{d}t$$
$$+ \int_0^T \left(\left(\mu(t)\alpha(t) + \rho(t)\alpha^2(t) + \rho(t)\alpha'(t)\right) v(t), \partial_t v(t)\right)\mathrm{d}t$$
$$+ \int_0^T (k(t)\Delta v(t), \Delta\partial_t v(t))\,\mathrm{d}t$$
$$-\int_0^T (T_r(t)\Delta v(t), \partial_t v(t))\,\mathrm{d}t = \int_0^T (f, \partial_t v(t))\,\mathrm{d}t = (f, v(T) - v(0)) = 0$$

- Grönwall's lemma cannot be applied!

Proof uniqueness ISP III

- The first four terms in the LHS can be handled as follows:

$$\int_0^T (\rho(t)\partial_{tt}v(t), \partial_t v(t))\,\mathrm{d}t = \tfrac{1}{2}\left\|\sqrt{\rho(T)}\partial_t v(T)\right\|^2 - \tfrac{1}{2}\int_0^T \left(\partial_t\rho, (\partial_t v)^2\right) \geqslant 0,$$

$$\int_0^T ((\mu + 2\rho\alpha)\,\partial_t v(t), \partial_t v(t))\,\mathrm{d}t \overset{\alpha\geqslant 0}{\geqslant} \mu_0 \int_0^T \|\partial_t v\|^2\,,$$

$$\int_0^T \left(\left(\mu\alpha + \rho\alpha^2 + \rho\alpha'\right) v(t), \partial_t v(t)\right)\mathrm{d}t = -\tfrac{1}{2}\int_0^T \left(\partial_t(\mu\alpha + \rho\alpha^2 + \rho\alpha'), v^2\right) \geqslant 0,$$

$$\int_0^T (k(t)\Delta v(t), \Delta\partial_t v(t))\,\mathrm{d}t = -\tfrac{1}{2}\int_0^T \left(\partial_t k, (\Delta v)^2\right) + \tfrac{1}{2}\left\|\sqrt{k(T)}\Delta v(T)\right\|^2 \geqslant 0$$

- The traction term is the most tricky one to handle

Proof uniqueness ISP IV

- If T_r is solely time dependent, then

$$-\int_0^T T_r(t)\left(\Delta v(t), \partial_t v(t)\right) \mathrm{d}t = \frac{1}{2}\int_0^T T_r(t)\int_\Omega \partial_t \left|\nabla v(t)\right|^2 \mathrm{d}t$$
$$= -\frac{1}{2}\int_0^T T_r'(t)\left\|\nabla v(t)\right\|^2 \mathrm{d}t \geqslant 0$$

- We get that

$$0 \leqslant \mu_0 \int_0^T \left\|\partial_t v(t)\right\|^2 \mathrm{d}t \leqslant 0$$

Therefore, $v = 0$ a.e. in Q_T

- Substituting $v = 0$ in (7) gives

$$(f, \varphi) = 0, \qquad \forall \varphi \in V.$$

We conclude by [Zeidler, 1990, Proposition 18.2] that $f = 0$ in $\mathrm{L}^2(\Omega)$

$h(t)$ is changing sign

- Problem:

$$
\begin{cases}
u_{tt} + u_t + u_{xxxx} - u_{xx} &= h(t)f(x) \quad & (x,t) \in (0,\pi)^2, \\
u(0,t) = u(\pi,t) &= 0 & t \in (0,\pi), \\
u_{xx}(0,t) = u_{xx}(\pi,t) &= 0 & t \in (0,\pi), \\
u(x,0) &= 0 & x \in (0,\pi), \\
u_t(x,0) &= 0 & x \in (0,\pi), \\
u(x,\pi) &= 0 & x \in (0,\pi),
\end{cases}
$$

  where $h(t) = (t+1)\sin(t) + (t+2)\cos(t)$ in $[0,\pi]$
- Solutions: next to $(u,f) = (0,0)$, also

$$
u(x,t) = \sin(x)\sin(t)t, \\
f(x) = \sin(x)
$$

$\alpha(t)$ is changing sign I

- Problem:

$$\begin{cases} u_{tt} + u_t + u_{xxxx} - u_{xx} & = h(t)f(x) & (x,t) \in (0,\pi) \times (0,4), \\ u(0,t) = u(\pi,t) & = 0 & t \in (0,4), \\ u_{xx}(0,t) = u_{xx}(\pi,t) & = 0 & t \in (0,4), \\ u(x,0) & = 0 & x \in (0,\pi), \\ u_t(x,0) & = 0 & x \in (0,\pi), \\ u(x,4) & = 0 & x \in (0,\pi), \end{cases}$$

where $h(t) = A + t\cos(t) > 0$ in $[0,4]$ with

$$A := \frac{2\exp(-2)\sin\left(2\sqrt{7}\right)\sqrt{7} + 14\exp(-2)\cos\left(2\sqrt{7}\right) + 14\cos(4) + 21\sin(4)}{-7 + \exp(-2)\sin\left(2\sqrt{7}\right)\sqrt{7} + 7\exp(-2)\cos\left(2\sqrt{7}\right)}$$

$\alpha(t)$ is changing sign II

- Solutions: next to $(u, f) = (0, 0)$, also

$$
\begin{aligned}
f(x) &= \sin(x), \\
u(x,t) &= f(x)\Phi(t),
\end{aligned}
$$

with

$$
\begin{aligned}
\Phi(t) = -\frac{1}{14}\exp\left(-\frac{t}{2}\right)\sin\left(\frac{\sqrt{7}}{2}t\right)(-2+A)\sqrt{7} \\
+\exp\left(-\frac{t}{2}\right)\cos\left(\frac{\sqrt{7}}{2}t\right)\left(1-\frac{A}{2}\right) \\
+\tfrac{1}{2}\,(t-2)\cos(t)+\tfrac{1}{2}\,(t-1)\sin(t)+\frac{A}{2}
\end{aligned}
$$

$T_r'(t)$ is changing sign

- Problem:

$$\begin{cases} u_{tt} + u_t + u_{xxxx} - \dfrac{50}{\sin(t)} u_{xx} &= h(t)f(x) \quad (x,t) \in (0,\pi)^2, \\ u(0,t) = u(\pi,t) &= 0 \qquad t \in (0,\pi), \\ u_{xx}(0,t) = u_{xx}(\pi,t) &= 0 \qquad t \in (0,\pi), \\ u(x,0) &= 0 \qquad x \in (0,\pi), \\ u_t(x,0) &= 0 \qquad x \in (0,\pi), \\ u(x,\pi) &= 0 \qquad x \in (0,\pi), \end{cases}$$

where $h(t) = 50t + (t+2)\cos(t) + \sin(t) > 0$, $\alpha(t) \geqslant 0$ and $\left(\alpha + \alpha^2 + \alpha'\right)' \leqslant 0$ in $[0,\pi]$

- Solutions: besides $(u,f) = (0,0)$, also

$$u(x,t) = \sin(x)\sin(t)t,$$
$$f(x) = \sin(x)$$

Principle of linear superposition

- v_{**} is the unique solution (see Theorem 1) to (5)-(6) when $f = 0$
- Solving problem (2)-(3) is equivalent with solving

$$\rho\partial_{tt}v_* + (\mu + 2\rho\alpha)\,\partial_t v_* + \left(\mu\alpha + \rho\alpha^2 + \rho\alpha'\right) v_* + \Delta(k\Delta v_*) - T_r\Delta v_* = f(\mathbf{x}), \quad (8)$$

with

$$\begin{cases} v_* = 0 & \text{on } \Sigma_T, \\ k\Delta v_* = 0 & \text{on } \Sigma_T, \\ v_*(\mathbf{x},0) = 0 & \mathbf{x} \in \Omega, \\ \partial_t v_*(\mathbf{x},0) = 0 & \mathbf{x} \in \Omega, \end{cases} \quad (9)$$

and

$$v_*(\cdot, T) = \frac{\xi_T(\cdot)}{h(T)} - v_{**}(\cdot, T) =: \widetilde{\xi}_T(\cdot)$$

- Landweber-Fridman iterative regularization method [Landweber, 1951, Fridman, 1956]
- Define the input-output operator $M_T \in \mathcal{L}\left(\mathrm{L}^2(\Omega), \mathrm{L}^2(\Omega)\right)$ by

$$M_T f = v(\cdot, T),$$

with $v \in \mathrm{C}\left([0, T], V\right)$ the unique solution to (8)-(9) for given f

- Finding a solution to the ISP is equivalent to solving

$$M_T f = \widetilde{\xi}_T,$$

or equivalent to solving the fixed point equation

$$f = f + \omega M_T \left(\widetilde{\xi}_T - M_T f\right), \qquad \omega > 0$$

- Method of successive approximations ($k \in \mathbb{N}$):

$$f_k := f_{k-1} - \omega M_T \left(M_T f_{k-1} - \widetilde{\xi}_T\right),$$

with an initial guess $f_0 \in \mathrm{L}^2(\Omega)$

Stopping criterion

- Error in the additional measurement, i.e.

$$\|\xi_T - \xi_T^e\| \leqslant e,$$

where $e(\tilde{e})$ depends on the noise level with magnitude $\tilde{e} > 0$
- Thus also $\widetilde{\xi}_T$ is perturbed, denote by $\widetilde{\xi}_T^e$
- The functions f_k^e and v_k^e are obtained by using the algorithm
- Discrepancy principle by Morozov [Morozov, 1966]: finish the iterations at the smallest index $k = k(e, \omega)$ for which

$$E_k := \left\| v_k^e(\cdot, T) - \widetilde{\xi}_T^e \right\| \leqslant \tau_0 e,$$

for some $\tau_0 > 1$ (typically between 1 and 1.2)

(i) Determine v_{**} and the final overdetermination $\widetilde{\xi}^e_T$;

(ii) Initial guess $f_0 \in \mathrm{L}^2(\Omega)$. Let v_0 be the solution to (8)-(9) with $f = f_0$;

(iii) Let

$$\omega_k = \frac{1}{\|f_k\|} \text{ for } k \geqslant 1, \quad \omega_0 = 1;$$

(iv) Assume that f_{k-1} and v_{k-1} have been constructed. Let w_{k-1} solve (8)-(9) with $f(x) = v_{k-1}(x, T) - \widetilde{\xi}^e_T(x)$;

(v) Define

$$f_k(x) = f_{k-1}(x) - \omega_{k-1} w_{k-1}(x, T), \qquad \text{a.a. } x \in \Omega,$$

and let v_k solve (8)-(9) with $f = f_k$. Then,
if $E_{k-1} > E_k$:
$k = k + 1$
else:
repeat step (v) with $\omega_{k-1} = \frac{\omega_{k-1}}{2}$;

(vi) Repeat steps (iv) and (v) until
- $E_k \leqslant \tau_0 e$, a maximum number of iterations is reached, $w_k < 1 \times 10^{-12}$;

(vii) Suppose that the algorithm is stopped after $\tilde{k}$ iterations with corresponding solution $\langle v_{\tilde{k}}, f_{\tilde{k}} \rangle$. Then, the approximating solution to the original problem (2)-(3) is given by $\langle h(v_{**} + v_{\tilde{k}}), f_{\tilde{k}} \rangle$.

Numerical experiment: setting

- A simply supported Euler-Bernoulli beam is considered
- $\Omega = [0, 1], \; T = 0.02$
- $\rho = \mu = k = T_r = 1, \; h(t) = 1$ and $\tilde{u}_0 = \tilde{v}_0 = 0$

Numerical experiment: setting

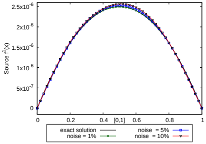
- Forward problems are discretized in time according to the backward Euler method with timestep 0.00001
- To solve the forward problems using Lagrange finite element basis functions, the equation is split into two second-order equations
- At each time step, the resulting elliptic mixed problems are solved numerically by the finite element method using first order (P1–FEM) Lagrange polynomials for the space discretization (the number of finite elements is taken to be equal to 200)
- A randomly generated uncorrelated noise is added to the additional condition in order to simulate the inherent errors present in real measurements (noise $\times\, \mathcal{N}(0, 1)$)
- FEniCS project — The finite element library DOLFIN [Logg and Wells, 2010, Logg et al., 2012b] from the FEniCS project [Logg et al., 2012a] is used

The exact sources used in the experiments are given by

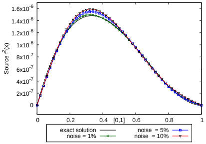
$$
\begin{aligned}
f^1(x) &= 0.00001 \times x(1-x),\\
f^2(x) &= 0.00001 \times x(x-1)^2,\\
f^3(x) &= \frac{0.00001}{2\pi} \times \sin(2\pi x),\\
f^4(x) &= 0.000001 \times \exp\left(-20(x-0.5)^2\right),\\
f^5(x) &= \begin{cases} 0 & 0 \leqslant x \leqslant \frac{1}{3} \\ 0.000006 \times \left(x - \frac{1}{3}\right) & \frac{1}{3} \leqslant x \leqslant \frac{1}{2} \\ -0.000006 \times \left(x - \frac{2}{3}\right) & \frac{1}{2} \leqslant x \leqslant \frac{2}{3} \\ 0 & \frac{2}{3} \leqslant x \leqslant 1 \end{cases}
\end{aligned}
$$

and

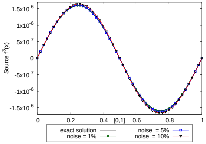
$$
f^6(x) = \begin{cases} 0 & 0 \leqslant x < \frac{1}{3} \\ 0.000001 & \frac{1}{3} \leqslant x \leqslant \frac{2}{3} \\ 0 & \frac{2}{3} < x \leqslant 1 \end{cases}
$$



(a)



(b)



(c)

Figure: The exact sources f^1, f^2 and f^3 and its corresponding numerical solution, retrieved using various levels of noise in the additional measurement (a,b,c).

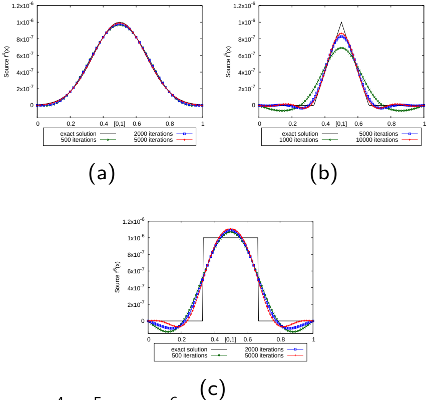
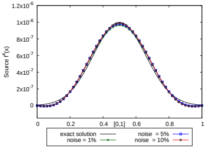
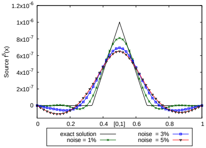


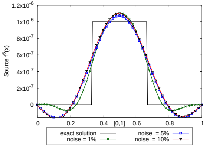
Figure: The exact sources f^4, f^5 and f^6 and its corresponding numerical solution, retrieved without noise on the measurement (a,b,c).



(a)



(b)



(c)

Figure: The exact sources f^4, f^5 and f^6 and its corresponding numerical solution, retrieved using various levels of noise in the additional measurement (a,b,c).

Table: The stopping iteration number $\tilde{k}$, the CPU time (mins) and the value of the relaxation parameter at $\tilde{k}$ for $\tilde{e} = 1\%$ and 5%.

$\tilde{e} = 1\%$	$\tilde{k}$	CPU time	$\omega_{\tilde{k}-1}$	$\tilde{e} = 5\%$	$\tilde{k}$	CPU time	$\omega_{\tilde{k}-1}$
f^1	149	170	546694.3	f^1	64	90	541650.8
f^2	221	242	1024892.8	f^2	123	145	1003726.2
f^3	192	212	885593.9	f^3	140	161	873533.3
f^4	98	123	1888012.6	f^4	63	91	1852594.9
f^5	4346	4183	250957.6	f^5	510	510	3332767.9
f^6	5000	4764	1835035.1	f^6	67	92	1834481.4

- The attainability of the stopping criterion becomes faster if $\tilde{e}$ increases
- Also for large noise level (e.g. $\tilde{e} = 5\%$), an accurate approximation for the sources is obtained
- The algorithm is more sensitive for increasing the amount of noise in the experiment
- Drawback: value relaxation parameter, long CPU time

Conclusions

- ISP associated with the dynamic vibration of a simply supported beam and plate was considered
- Also other boundary conditions can be considered
- Uniqueness of a solution to the IP is proved
  $\oplus$ without additional assumptions on the solution
- An adaptive Landweber-Fridman type iterative regularization method is used to obtain an approximation of the unknown load source
- The one-dimensional numerical experiments carried out were implemented using the FEM and validated the stability of the proposed iterative procedure
- Disadvantage: process is time consuming

Future research

- Validity of the numerical scheme?
- Nondimensionalization
- Comparison of the results with faster iterative methods such as the conjugate gradient method

M21: Inverse Problems in Science and Engineering

Inverse problems arise in many areas of mathematical physics and applications are rapidly expanding to geophysics, chemistry, medicine and engineering. This minisymposium focuses on both analytical and computational methods for inverse problems in Science and Engineering. The approaches developed for such problems generally include numerical approximations, stability analysis, proofs of uniqueness and/or existence of the solution.

The minisymposium aims at bringing together well established scientists as well as young researchers working on inverse problems for partial differential equations **to honour one of the experts in this field, Professor Marian Slodička, on the occasion of his 60th birthday**. The topics of the minisymposium range from the mathematical modelling and the theoretical analysis of inverse problems for partial differential equations where some parameters (right-hand side, kernel, diffusion coefficient, etc.), unknown boundary condition(s) or portion of the boundary are to be found, to the development of efficient numerical schemes and their practical implementations.

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