Subject for Master diploma work: Inverse Problems and Regularization

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Abstract

Inverse problems often lead to mathematical problems that are not well-posed in the sense of Hadamard, i.e. to ill-posed problems. This means especially that their solution is unstable under data perturbations. Numerical methods that can cope with this problem are so-called regularization methods. This thesis is devoted to the study of these regularization methods.

Details

A definition of an inverse problem can be found for instance in [1]:

Inverse problems are concerned with determining causes for a desired or an observed effect.

In literature, three main types of inverse problems are distinguished:

(a) parameter identification, where the material parameters appearing in the equation are not known and should be reconstructed, e.g., diffusion coefficients, source terms, etc.,

(b) boundary value inverse problems, where direct measurements on the bound- ary (or a part of it) are unfeasible and have to be determined,

(c) evolutionary inverse problems in which the initial conditions are not known and have to be reconstructed.

Inverse problems are inherently driven by applications and they arise in a vast variety of practical situations such as biomedical engineering, image processing and non-destructive material evaluation. Moreover, inverse problems are often ill-posed in the sense of Hadamard. This means that there is either no solution in a classical sense or if there is any, then it might not be unique or might not depend continuously on the data.

In an abstract setting, an inverse problem is defined by

Solve the equation $F(x) = y$ for $x \in X$ (Banach, Hilbert,... space) when the data $y \in Y$ (Banach, Hilbert,... space) is given and where F^{-1} does not exist or is not continuous (the operator F is related with the forward problem).

We try to solve $F(x) = y$ when the generalized solution $x^* = F^{-1}(y)$ does not exist. The generalized solution is the true solution or minimal norm solution. Usually, we do not have the exact data $y = F(x^*)$ but only noisy data y_δ ($= F(x^*) + \text{noise}$). The question is: how can we solve $F(x) = y_\delta$

in some approximate sense? An answer can be given by the so-called regularization methods.

The main tasks of this master thesis are

- To give an introduction on inverse (ill-posed) problems;
- To study theoretically regularization methods as Tikhonov regularization and Landweber regularization. A comparison with the conjugate gradient method can be made.
- To solve an inverse problem (for instance the determination of a space-dependent heat source) by both Tikhonov and Landweber regularization by using for instance the finite element library DOLFIN from the FEniCs project. The goal is to investigate the advantages and disadvantages of both methods.

The book by Engl can be used as reference work [1].

References

[1] H. W. Engl, M. Hanke, A. Neubauer, Regularization of Inverse Problems, Vol. 375 of Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, 1996.