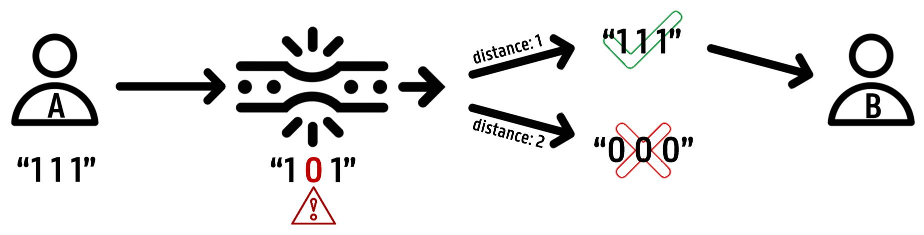
FACULTY OF SCIENCES

SMALL WEIGHT CODE WORDS

arising from the incidence matrix of points and hyperplanes

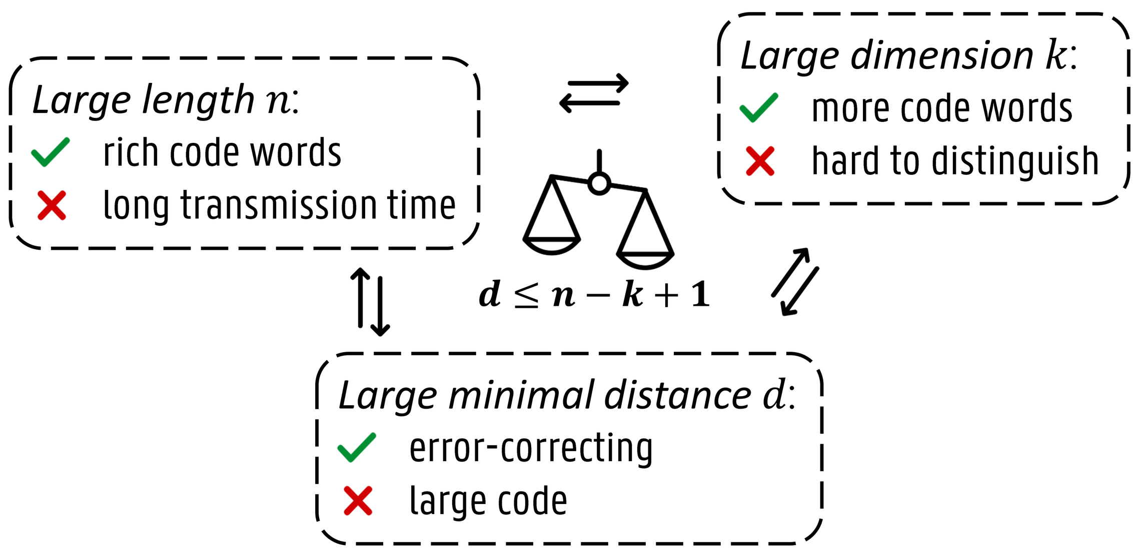
Root of communication

Code words have been used throughout the ages to **transmit a message from point A to point B**. During this transmission, a code word can alter and errors can occur due to external factors (e.g. a hardware defect).
Making a good choice of which code to use and what (de)coding methods to apply, error-correction can prevent the loss of information.



Linear codes

A $[n,k,d]$-**linear code** over the field F_q is a subspace of the vector space $V(n,q)$. Its vectors are called **code words**.



Distance and weight

The (Hamming) **distance** between two code words is measured as the number of different digits arising in both vectors.
The **weight** of a code word is the number of non-zero digits it contains.

The **minimal distance** $\boldsymbol{d}$ of a code is the smallest distance between two different code words. For linear codes, this is equal to the **minimal weight** of the code.

Coding theory

Coding theory analyses codes, their properties and usage in various applications. The study of 'good' error-correcting codes, like mentioned above, is an important topic within this field of study.

The code $C_{n-1}(n,q)$

Numbering the points and hyperplanes of the projective space $PG(n,q)$, we can compose the incidence matrix between these two types of spaces.
The **code** $\boldsymbol{C_{n-1}(n,q)}$ is equal to the span of the rows of this incidence matrix. As such, each code word can be interpreted as **a 'linear combination' of hyperplanes** in $PG(n,q)$.

$$\begin{pmatrix} 1&1&1&0&0&0&0\\ 1&0&0&1&1&0&0\\ 0&0&1&1&0&1&0\\ 0&1&0&0&1&1&0\\ 1&0&0&0&0&1&1\\ 0&1&0&1&0&0&1\\ 0&0&1&0&1&0&1 \end{pmatrix}$$

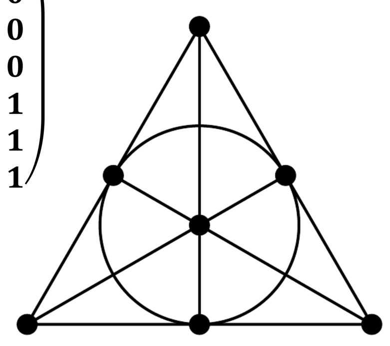
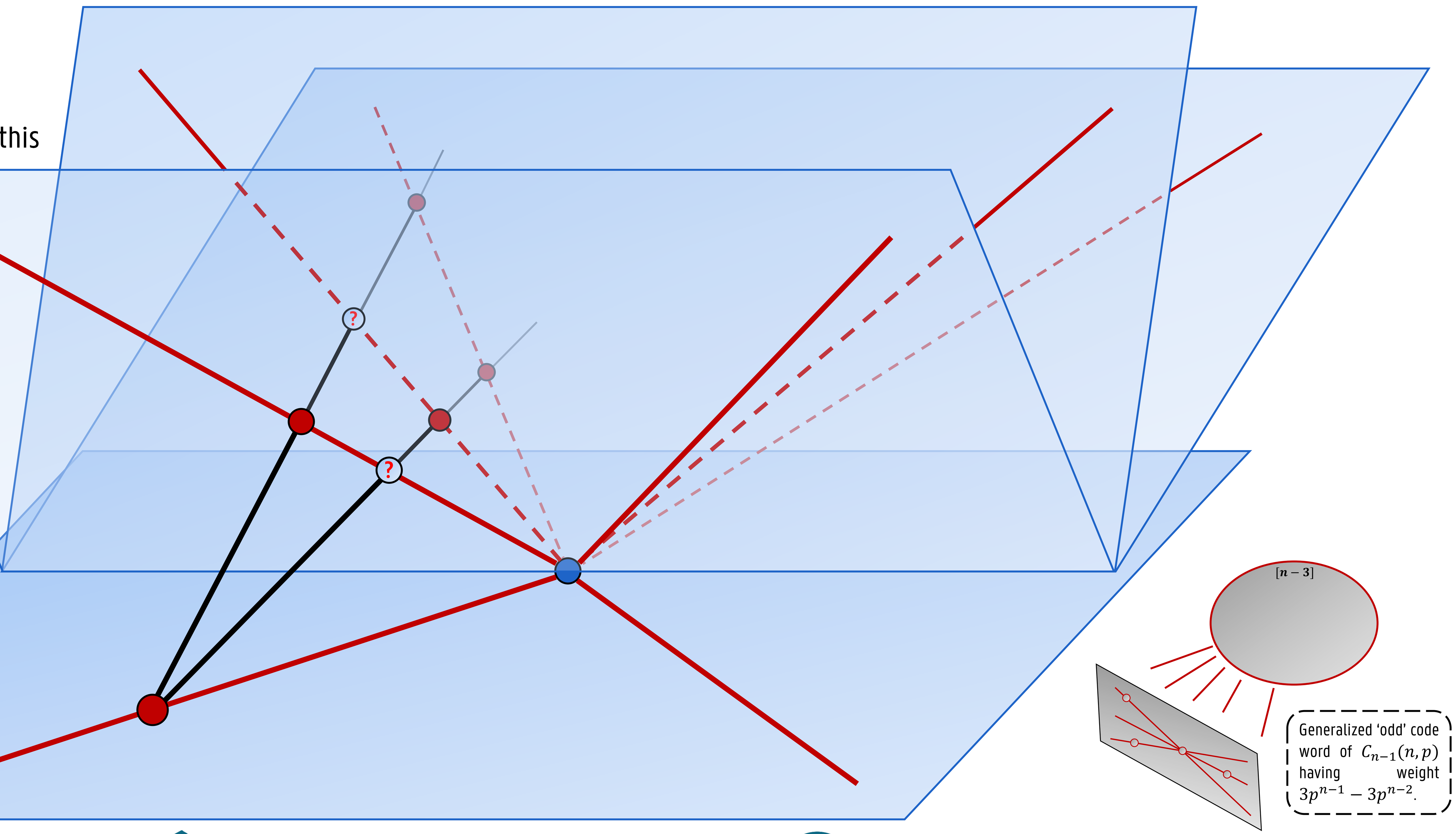


Figure explanation

The **figure** occupying the center of this poster is an illustration of how we proved that each small weight code word of the code $\boldsymbol{C_2(3,p)}$ is always associated with a linear combination of at most two (hyper)planes.
We can 'force' the two **black** lines into the same plane of all red lines at the left side, by knowing they have to intersect red lines when they intersect the corresponding plane in a point of the code word.



Results for the plane

Below are some results concerning $\boldsymbol{C_1(2,q)}$.
For $\boldsymbol{q=p}$ prime, all code words of weight at most $O(3p)$ are associated with a linear combination of at most two lines.
For $\boldsymbol{q}$ not prime, all code words of weight at most $O(\frac{1}{2}q^{1.5})$ are associated with a linear combination of at most $O(\frac{1}{2}\sqrt{q})$ lines.

Results for arbitrary dimension

It is a well known result that the minimum weight of $\boldsymbol{C_{n-1}(n,q)}$ equals the number of points in a hyperplane (= $\boldsymbol{\theta_{n-1}}$).
Recently, it was proved that the code words having second smallest weight (namely $\boldsymbol{2q^{n-1}}$) are associated with **the symmetric difference of two different hyperplanes**.

An 'odd' code word of $C_1(2,p)$

Contrary to our intuition, one can construct a code word of weight $3p-3$, **not associated with a linear combination of three lines**.
Generalizing this result, we deduce that small weight code words do not always arise from linear combinations of few hyperplanes, in contrary to speculation.

The third smallest weight

Our main result states that the **third smallest weight** code words of $C_{n-1}(n,q)$ have weight $\boldsymbol{2q^{n-1}+\theta_{n-2}}$. Such code words are associated with a linear combination (not the difference) of two different hyperplanes.

The fourth smallest weight?

We have deduced **several lower bounds** for the fourth smallest weight $\boldsymbol{w_4}$ in the code $C_{n-1}(n,q)$, depending on the value q:

- $w_4 \geq O(\frac{5}{2}q^{n-1})$ when q is prime.
- $w_4 \geq O(\frac{1}{2}q^{n-1}\sqrt{q})$ otherwise.

Future work

We will try to **optimise** our lower bounds and **classify** the small weight code words of $C_{n-1}(n,p)$ as being either linear combinations of hyperplanes or generalizations of the 'odd' code word mentioned above.

GHENT UNIVERSITY

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