Combinatorics 2022

Higgledy-piggledy sets

in projective spaces

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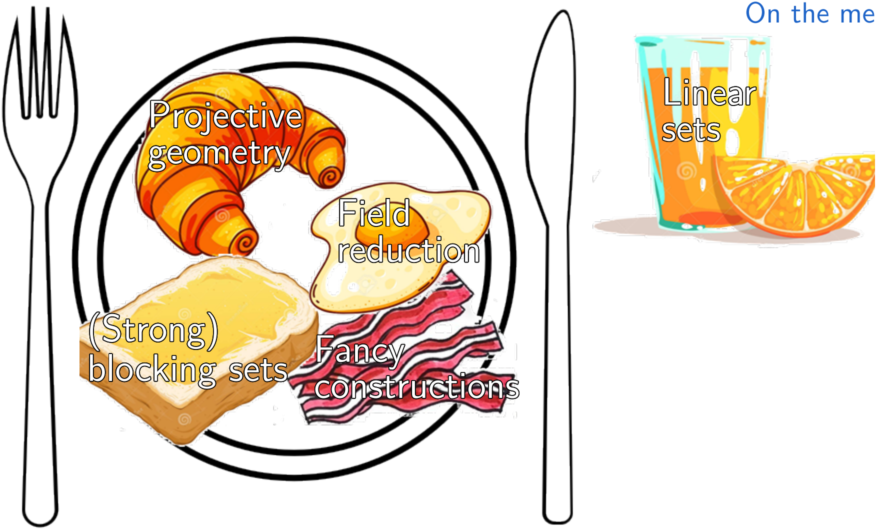
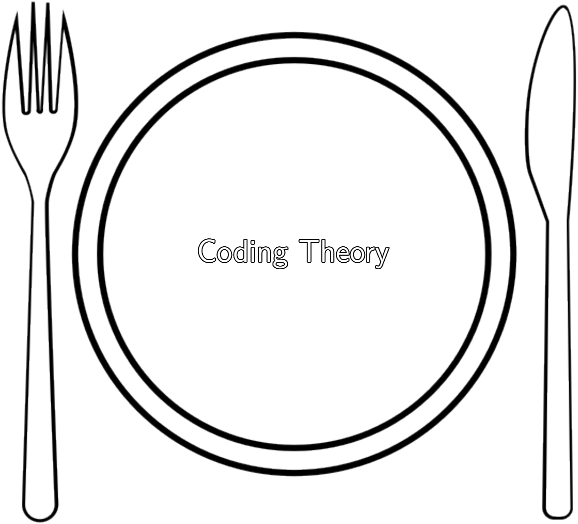
GHENT UNIVERSITY

Overview

On the menu


Projective geometry
Field reduction
(Strong) blocking sets
Fancy constructions
Linear sets

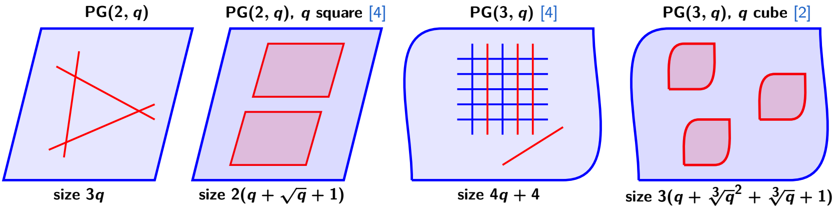
On the menu



Strong blocking sets

Let $N \in \mathbb{N}^{\times}$ and q be a prime power; take $k \in \{0, \ldots, N-1\}$.

A *strong k-blocking set* of $\mathrm{PG}(N, q)$ is a point set that meets every $(N-k)$-dimensional space κ in a point set spanning κ.



A *higgledy-piggledy set of k-spaces* is a set $\mathcal{K}$ of k-spaces such that the point set $\cup\mathcal{K}$ is a strong k-blocking set.

Motivation

Let $n \in \mathbb{N}^{\times}$ and $k' \in \{0, 1, \ldots, n\}$.

Consider (q-ary) linear $[n, k']_q$-code $\mathcal{C}$ (k'-space of $\mathrm{V}(n, q)$).

- Elements/vectors of $\mathcal{C}$ are called *codewords*.
- The *support* $\mathrm{supp}(c)$ of a codeword $c \in \mathcal{C}$ is the subset of $\{1, 2, \ldots, n\}$ of non-zero positions in c.

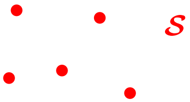
A codeword $c \in \mathcal{C}$ is called *minimal* if

$$\forall c' \in \mathcal{C} : \mathrm{supp}(c') \subseteq \mathrm{supp}(c) \Rightarrow c' \in \langle c \rangle_{\mathbb{F}_q}.$$

A linear code is called *minimal* if all its codewords are minimal.

Motivation

Correspondence with minimal codes



$\mathcal{S}$ strong block. set w.r.t. hyperplanes, $\mathcal{S} := \{P_1, \dots, P_{|\mathcal{S}|}\}$.

$$\begin{matrix} P_1 & P_2 & P_3 & & P_i & & P_{|\mathcal{S}|} \end{matrix}$$
$$\begin{pmatrix} x_{10} & x_{20} & x_{30} & \cdots & x_{i0} & \cdots & x_{|\mathcal{S}|0} \\ x_{11} & x_{21} & x_{31} & \cdots & x_{i1} & \cdots & x_{|\mathcal{S}|1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1N} & x_{2N} & x_{3N} & \cdots & x_{iN} & \cdots & x_{|\mathcal{S}|N} \end{pmatrix}$$

coordinates of P_i

Theorem (G. N. Alfarano, M. Borello, A. Neri [1]; C. Tang, Y. Qiu, Q. Liao, Z. Zhou [8]).
→ the generator matrix of a **minimal** linear $[|\mathcal{S}|, N+1]_q$-code!

Goal: Finding small ~~strong blocking sets~~ higgledy-piggledy sets.

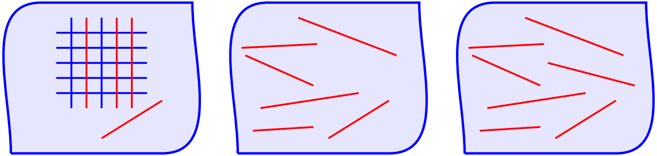
3 Known results

The line case $(k = 1)$

Theorem (Fancsali, Sziklai [5]; T. Héger, Z. L. Nagy [7]).
If $q \geqslant N + \lfloor \frac{N}{2} \rfloor$, then
A higg.-pigg. line set contains at least $N + \lfloor \frac{N}{2} \rfloor - \lfloor \frac{N-1}{q} \rfloor$ lines.

Theorem (Fancsali, Sziklai [5]; T. Héger, Z. L. Nagy [7]).
If $q \geqslant 2N - 1$, then
There exists a higg.-pigg. line set of asymptotic size $2N - 1$.

PG(3, q) [4] | **PG(4, q)**, $q > 36086$, $\mathrm{char}(q) \neq 2, 3$ [3] | **PG(5, q)** [2]



size $4q + 4$ | size $6q + 6$ | size $7q + 7$

Known results

Theorem (Sz. L. Fancsali, P. Sziklai [6]; LD).
A higgledy-piggledy set of k-spaces in $\mathrm{PG}(N,q)$ contains more than

$$\min\left\{q, \max\left\{\sum_{i=0}^{k}\left\lfloor\frac{N-k+i}{i+1}\right\rfloor, \sum_{i=1}^{N-k}\left\lfloor\frac{k+i}{i}\right\rfloor\right\}\right\}$$

elements.

Theorem (Fancsali, Sziklai [6]; T. Héger, Z. L. Nagy [7]).
If $q > N+1$, then
There exists a higg.-pigg. set of k-spaces of asymptotic size $(N-k)(k+1)+1$.

Construction methods

Theorem (Sz. L. Fancsali, P. Sziklai [6]).
A set $\mathcal{K}$ of k-spaces, $|\mathcal{K}| \leqslant q$, is a higg.-pigg. set
$\Leftrightarrow$ no $(N-k-1)$-space meets all its elements.

Projection
eg. consider one line parity case.

Dualisation
$k \to N-k-1$ (lower bound).

Field reduction
only if $N+1$ is composite.

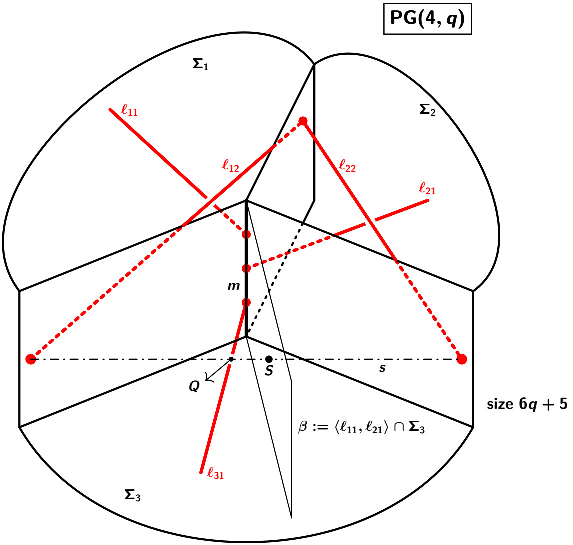
Coordinates
disjoint six lines example in $\mathrm{PG}(4, q)$.

Probability
improvements on lower/upper bounds.

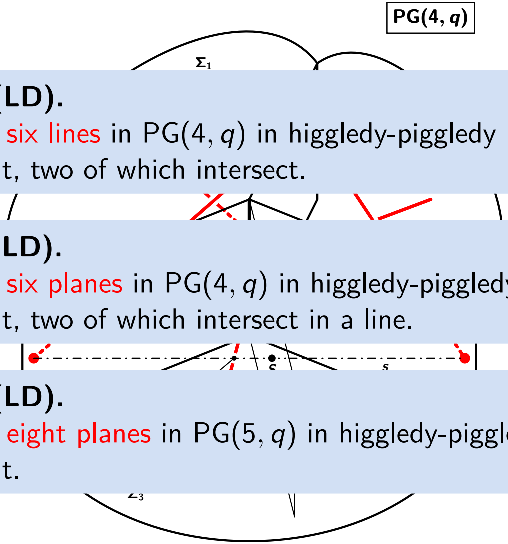
Elem. geometry
intersecting six lines, see next slide.

Construction methods

An example



An example



Theorem (LD).
There exist six lines in PG$(4, q)$ in higgledy-piggledy arrangement, two of which intersect.

Corollary (LD).
There exist six planes in PG$(4, q)$ in higgledy-piggledy arrangement, two of which intersect in a line.

Theorem (LD).
There exist eight planes in PG$(5, q)$ in higgledy-piggledy arrangement.

Conclusion

Open problems and possible future approaches

- Find less than $2N-1$ higg.-pigg. lines, towards $\left\lfloor \frac{3}{2}N \right\rfloor$?
- Find less than $(N-k)(k+1)+1$ higg.-pigg. k-spaces?
- Exploit the known classification results on linear sets to find small higg.-pigg. sets?
- Improve current probabilistic applications?

L. Denaux. Higgledy-piggledy sets in projective spaces of small dimension. *Electron. J. Comb.*, accepted. (arXiv:2109.08572)

Thank you for listening

Any questions?

Suggestions?

Marvellous revelations?

References

[1] **G. N. Alfarano, M. Borello, and A. Neri.** A geometric characterization of minimal codes and their asymptotic performance. *Adv. in Math. of Commun.*, 16(1):115–133, 2022.

[2] **D. Bartoli, A. Cossidente, G. Marino, and F. Pavese.** On cutting blocking sets and their codes. *Forum Math.*, 34(2):347–368, 2022.

[3] **D. Bartoli, G. Kiss, S. Marcugini, and F. Pambianco.** Resolving sets for higher dimensional projective spaces. *Finite Fields Appl.*, 67:101723, 14, 2020.

[4] **A. A. Davydov, M. Giulietti, S. Marcugini, and F. Pambianco.** Linear nonbinary covering codes and saturating sets in projective spaces. *Adv. Math. Commun.*, 5(1):119–147, 2011.

[5] **Sz. L. Fancsali and P. Sziklai.** Lines in higgledy-piggledy arrangement. *Electron. J. Combin.*, 21(2):Paper 2.56, 15, 2014.

[6] **Sz. L. Fancsali and P. Sziklai.** Higgledy-piggledy subspaces and uniform subspace designs. *Des. Codes Cryptogr.*, 79(3):625–645, 2016.

[7] **T. Héger and Z. L. Nagy.** Short minimal codes and covering codes via strong blocking sets in projective spaces. *IEEE Trans. Inform. Theory*, 68(2):881–890, 2022.

[8] **C. Tang, Y. Qiu, Q. Liao, and Z. Zhou.** Full characterization of minimal linear codes as cutting blocking sets. *IEEE Trans. Inform. Theory*, 67(6, part 2):3690–3700, 2021.