GHENT
UNIVERSITY

eSeminar UGent-VUB

Constructing saturating sets in projective spaces

using subgeometries

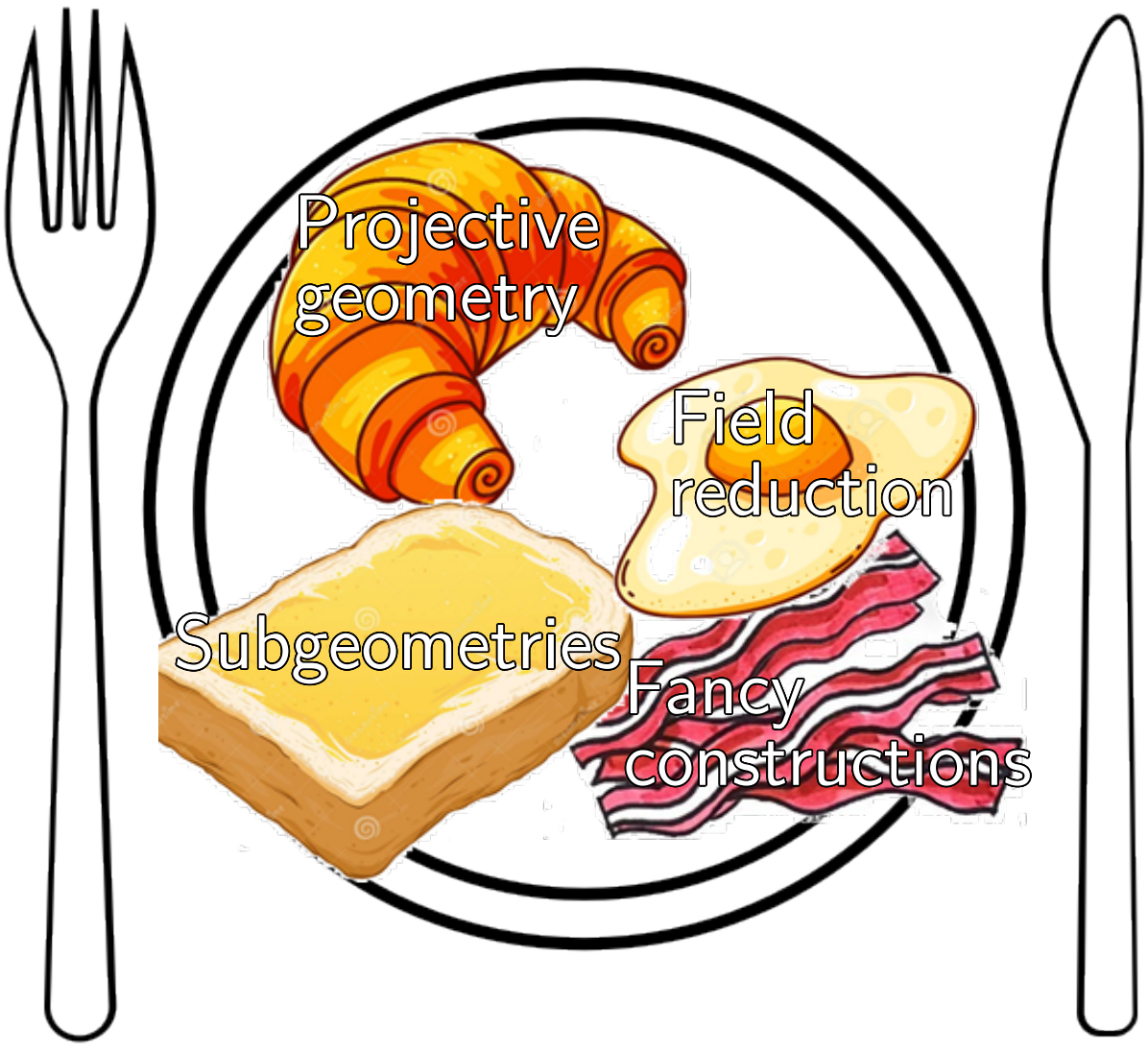
Lins Denaux

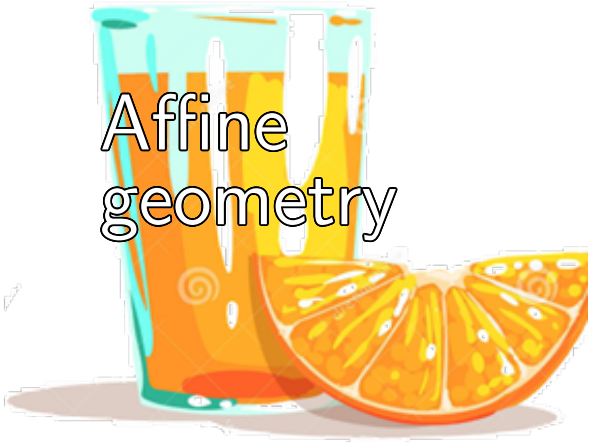
14th of January 2021

Overview

1 Introduction

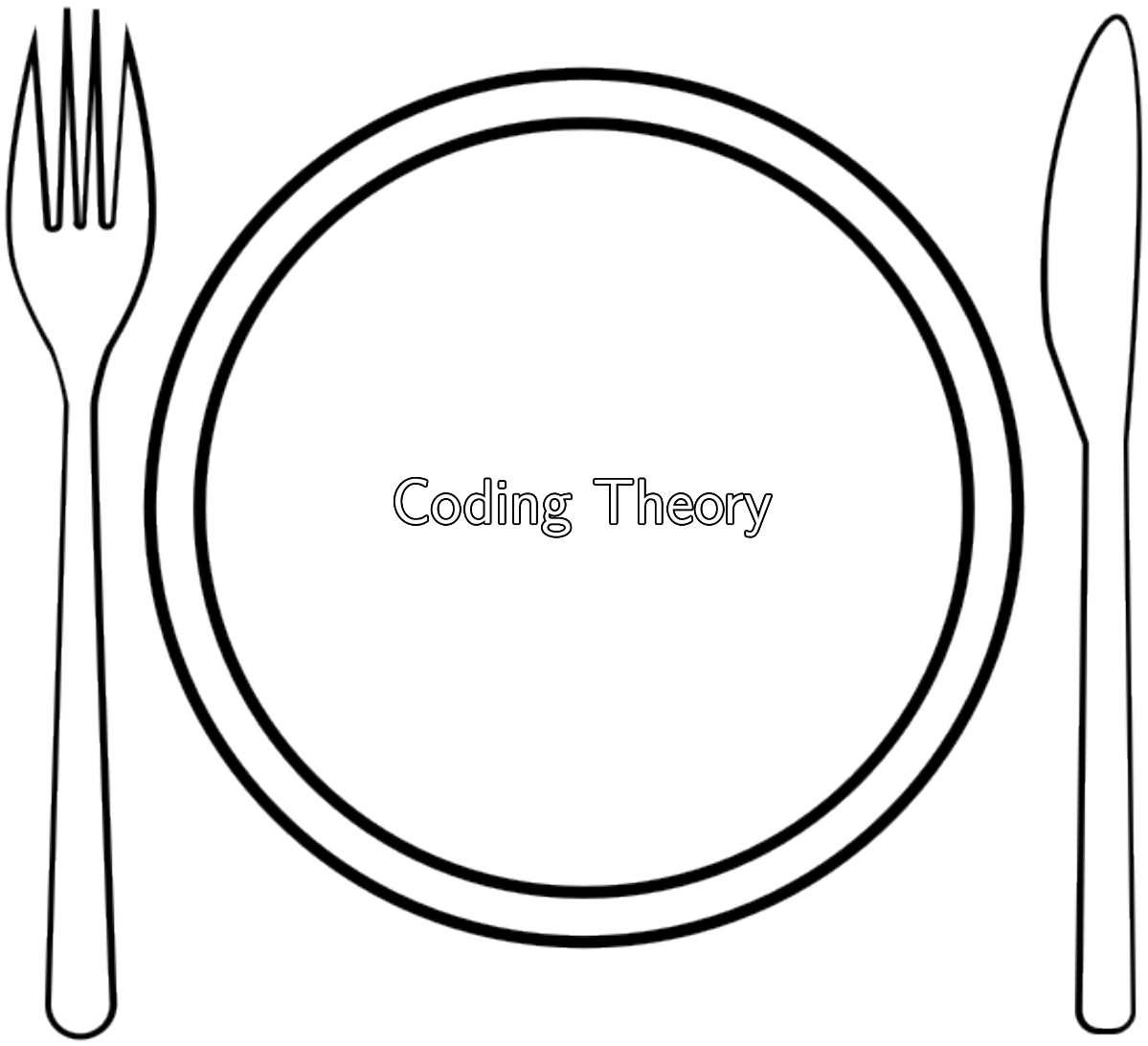
On the menu





1 Introduction

On the menu



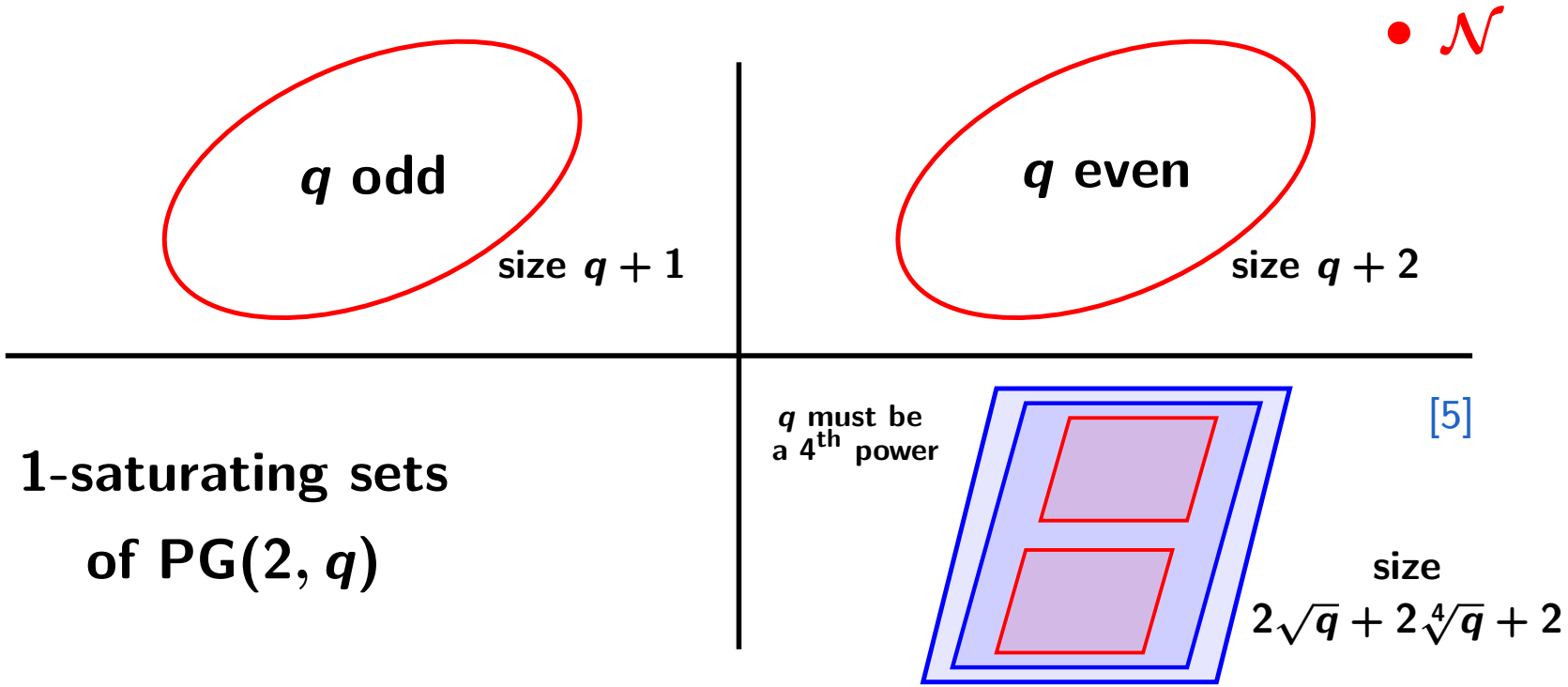
1 Introduction

Definition and examples

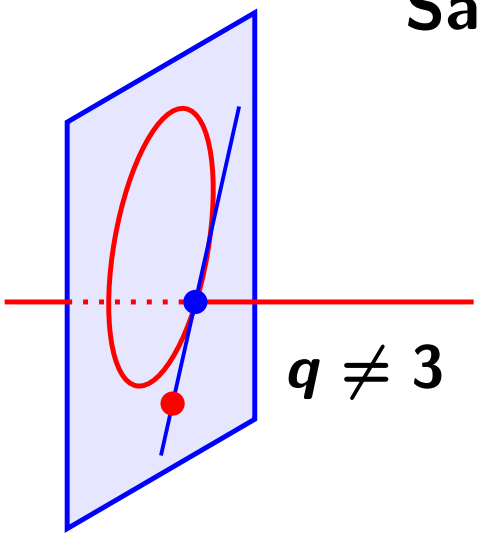
Let $n \in \mathbb{N}^{\times}$, q a prime power and $\varrho \in \{0, 1, \ldots, n\}$.

A ***ϱ-saturating set $\mathcal{S}$*** of $\mathrm{PG}(n, q)$: point set such that

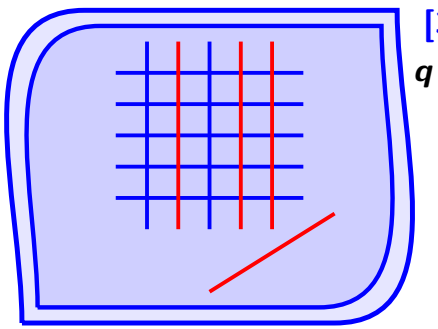
- any point of $\mathrm{PG}(n, q)$ lies in span of $\leqslant \varrho + 1$ points of $\mathcal{S}$.



Saturating sets of PG(3, q)



- **1**-saturating set of PG(3, q) [3].
- Size: $2q + 1$.



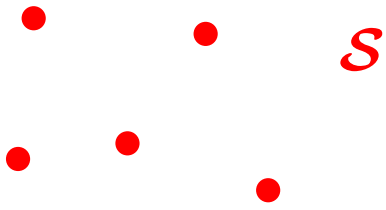
[3] q must be cube

- **2**-saturating set of PG(3, q) [5].
- Size: $4\sqrt[3]{q} + 4$.

$s_q(n, \varrho)$:= size of a smallest ϱ-saturating set of PG(n, q).

1 Introduction

Correspondence with covering codes



ϱ-saturating set of $\mathrm{PG}(n,q)$, $\mathcal{S} := \{P_1, P_2, P_3, \ldots, P_{|\mathcal{S}|}\}$.

$$\begin{array}{c} P_1 \quad P_2 \quad P_3 \qquad\quad P_i \qquad\quad P_{|\mathcal{S}|} \\ \begin{pmatrix} x_{10} & x_{20} & x_{30} & \cdots & x_{i0} & \cdots & x_{|\mathcal{S}|0} \\ x_{11} & x_{21} & x_{31} & \cdots & x_{i1} & \cdots & x_{|\mathcal{S}|1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1n} & x_{2n} & x_{3n} & \cdots & x_{in} & \cdots & x_{|\mathcal{S}|n} \end{pmatrix} \end{array}$$

coordinates of P_i

PC-matrix of a $[|\mathcal{S}|, |\mathcal{S}| - n - 1]_q$ $(\varrho+1)$-*covering code*!

Any vector of $\mathbb{F}_q^{|\mathcal{S}|}$ lies within Hamming distance $\varrho + 1$ of a codeword.

Goal: Finding good upper bounds for $s_q(n, \varrho)$.

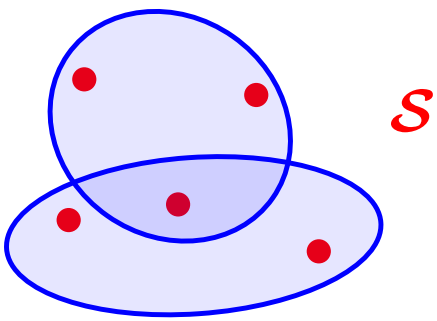
Introduction

A naive lower bound

Proposition (LD, 2020 [9])
Let $\mathcal{S}$ be a ϱ-saturating set of PG(n,q). Then

$$|\mathcal{S}| > \frac{\varrho+1}{e}\cdot q^{\frac{n-\varrho}{\varrho+1}} + \frac{\varrho}{2}.$$

Proof. $$\frac{(|\mathcal{S}|-\frac{\varrho}{2})^{\varrho+1}}{(\varrho+1)!}\theta_\varrho \geqslant \binom{|\mathcal{S}|}{\varrho+1}\theta_\varrho \geqslant \theta_n.$$ ■



Keep in mind

$$s_q(n,\varrho) \quad \gtrsim \quad \varrho\cdot q^{\frac{n-\varrho}{\varrho+1}}.$$

1 Introduction

The jungle of known results

PG$(2, q)$: *LOTS of research!*

- Strongly $\sim$ to complete caps.
- Often computer searches.
- Nice survey in [7]. (Davydov & Östergård, 2000)

Keep in mind

$$s_q(n, \varrho) \quad \gtrsim \quad \varrho \cdot q^{\frac{n-\varrho}{\varrho+1}}.$$

~~Hypothesis (LD, 2019)~~

Desperate wish (LD, 2020)

$$s_q(n, \varrho) \quad \lesssim \quad \varrho \cdot q^{\frac{n-\varrho}{\varrho+1}},$$

for all n, $\varrho \leqslant n$ and ∞-many q.

PG(n, q): *quite a lot of research*

- Davydov et al., 2011 [5]

$$s_q(n, \varrho) \lesssim \binom{n+1}{\varrho} q^{\frac{n-\varrho}{\varrho+1}}$$

if q is a $(\varrho+1)^{th}$ power.

- Bartoli et al., 2017, 2019 [1, 2]

$$s_q(n, 1) \lesssim 2q^{\frac{n-1}{2}}\sqrt{\ln(q)}$$

if n is even and n, q are large.

- A few more, see later.

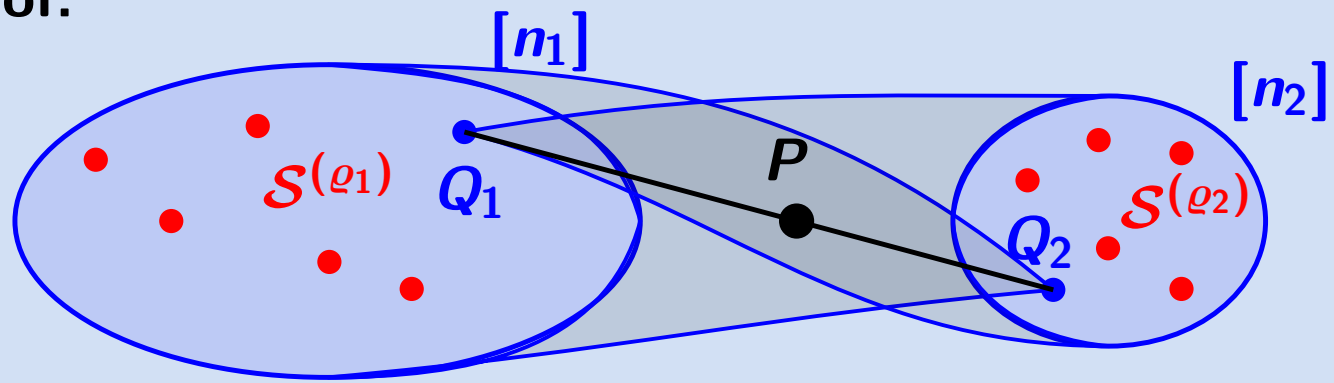
Empty results from the past: a flashback

An inductive fiasco

Theorem (Davydov & Östergård, 2000 [7])

$$s_q(n_1 + n_2 + 1, \varrho_1 + \varrho_2 + 1) \leqslant s_q(n_1, \varrho_1) + s_q(n_2, \varrho_2).$$

Proof.



Corollary

$$s_q(2\varrho + 1, \varrho) \leqslant (\varrho + 1)(q + 1).$$

Keep in mind

$$s_q(n, \varrho) \quad \gtrsim \quad \varrho \cdot q^{\frac{n-\varrho}{\varrho+1}}.$$

Empty results from the past: a flashback

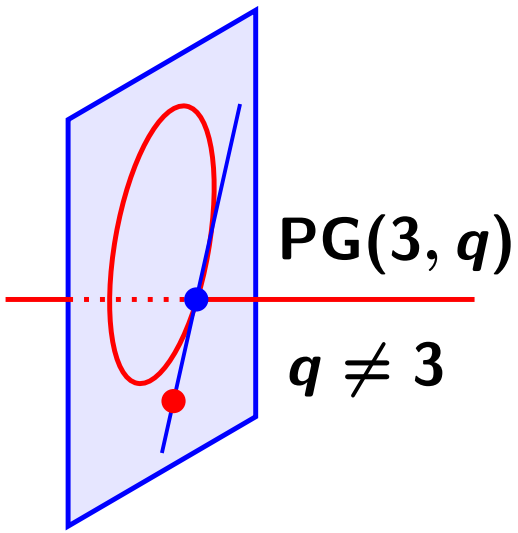
An inductive fiasco

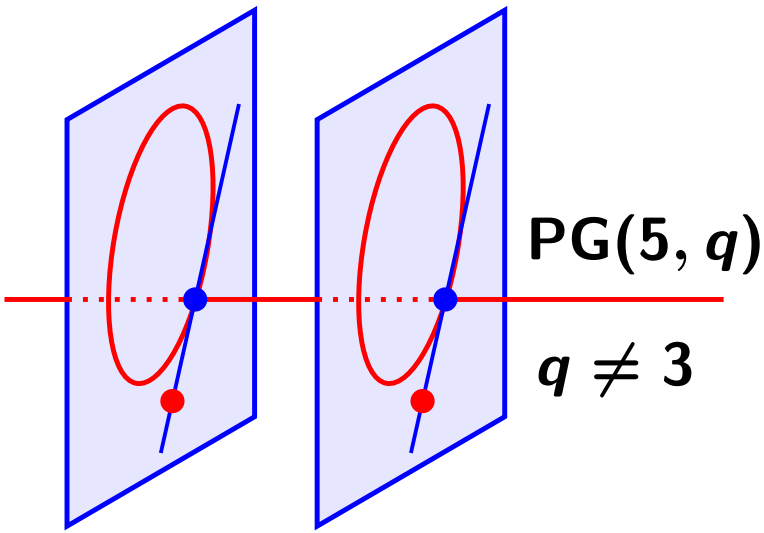
Corollary

$$s_q(2\varrho+1,\varrho) \leqslant (\varrho+1)(q+1).$$

Keep in mind

$$s_q(n,\varrho) \quad \gtrsim \quad \varrho \cdot q^{\frac{n-\varrho}{\varrho+1}}.$$





Theorem (Davydov [3])

$$s_q(3,1) \leqslant 2(q+1)-1.$$

Theorem (Davydov & Östergård, 2000 [7])

$$s_q(5,2) \leqslant 3(q+1)-2.$$

Empty results from the past: a flashback

An inductive fiasco

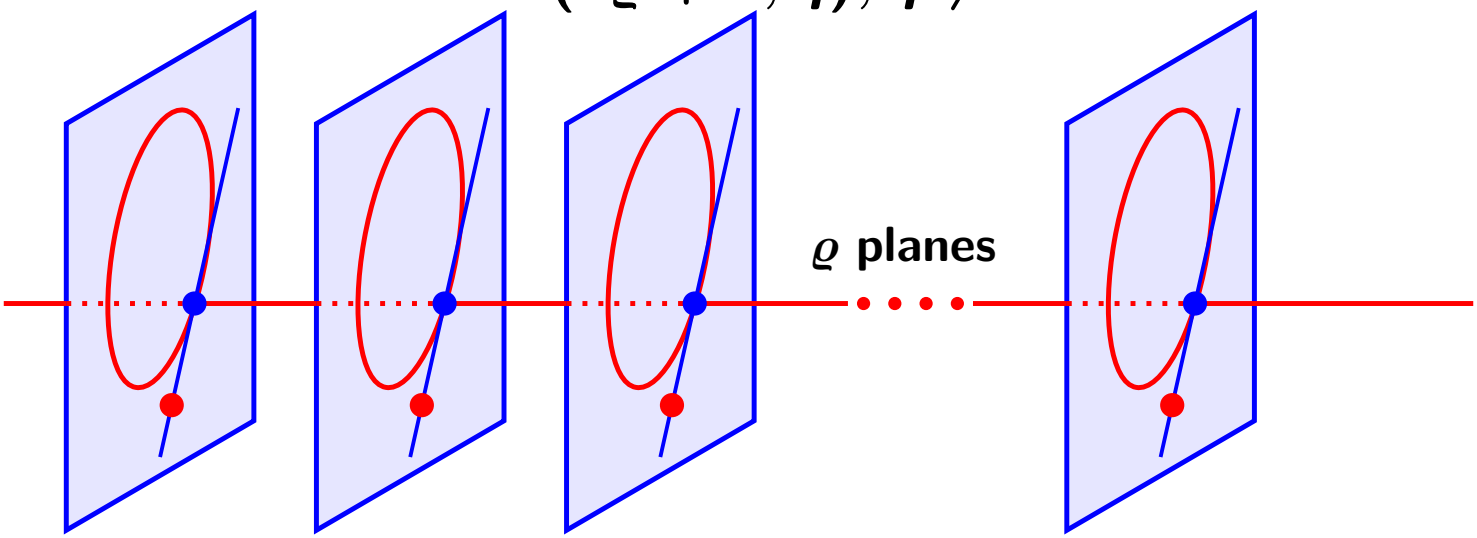
Corollary

$s_q(2\varrho+1,\varrho) \leqslant (\varrho+1)(q+1)$.

Keep in mind

$$s_q(n,\varrho) \quad \gtrsim \quad \varrho \cdot q^{\frac{n-\varrho}{\varrho+1}}.$$

PG$(2\varrho+1, q)$, $q \neq 3$



Theorem (LD, 2019)

If $q \neq 3$ and $\varrho \leqslant q$, then $s_q(2\varrho+1,\varrho) \leqslant (\varrho+1)(q+1)-\varrho$.

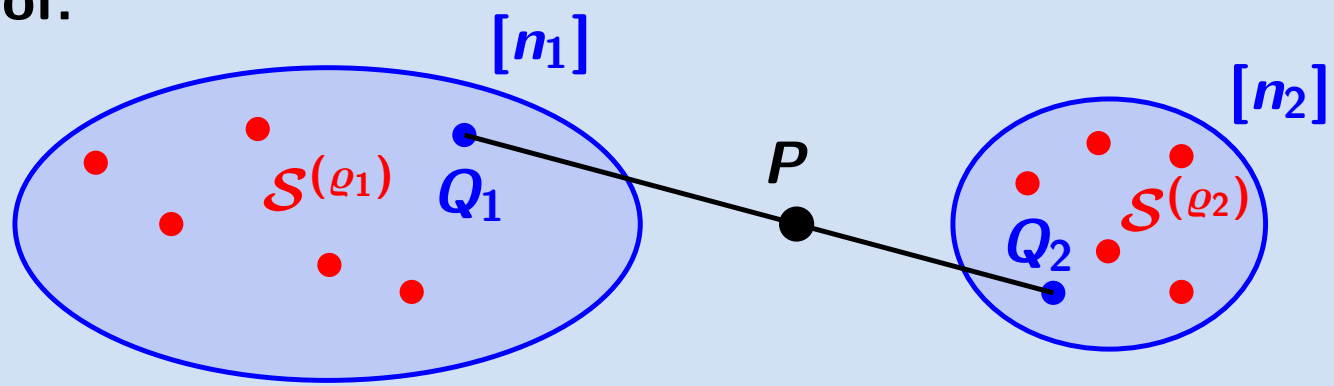
Empty results from the past: a flashback

An inductive fiasco

Theorem (Davydov & Östergård, 2000 [7])

$$s_q(n_1 + n_2 + 1, \varrho_1 + \varrho_2 + 1) \leqslant s_q(n_1, \varrho_1) + s_q(n_2, \varrho_2).$$

Proof.



Corollary

$$s_q(2\varrho + 1, \varrho) \leqslant (\varrho + 1)(q + 1).$$

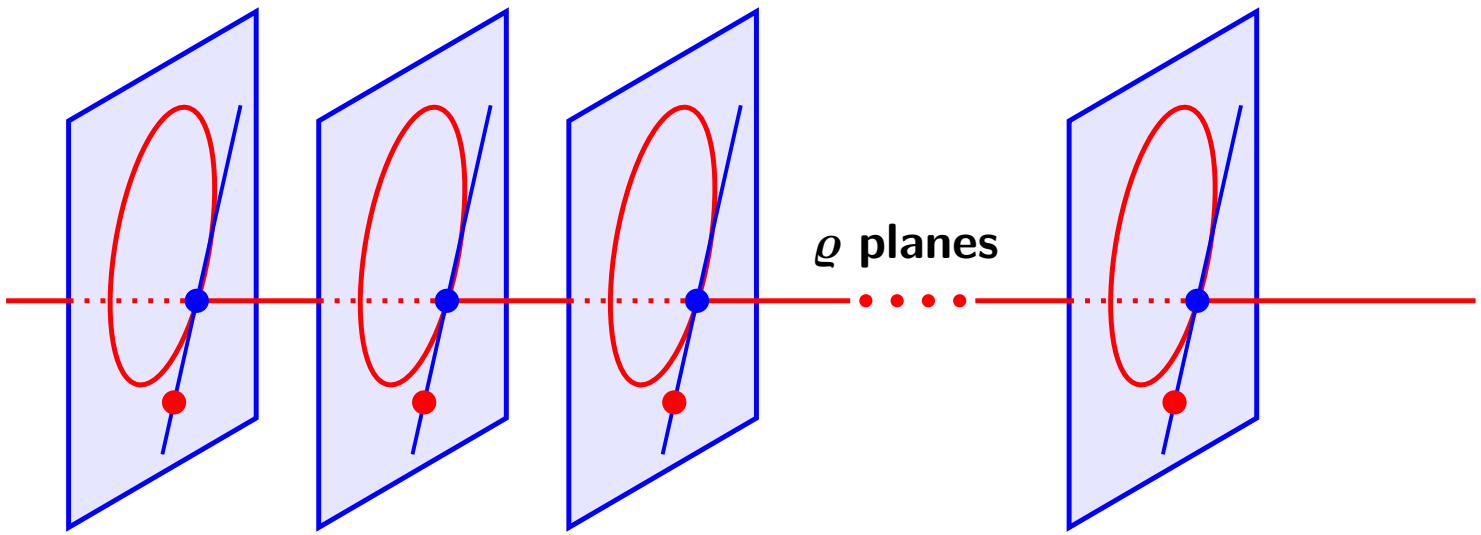
Corollary

$$s_q\Big(k(\varrho+1) + \varrho, \varrho\Big) \leqslant (\varrho+1)\theta_k.$$

Empty results from the past: a flashback

Corollary

$s_q(2\varrho+1,\varrho) \leqslant (\varrho+1)(q+1)$.



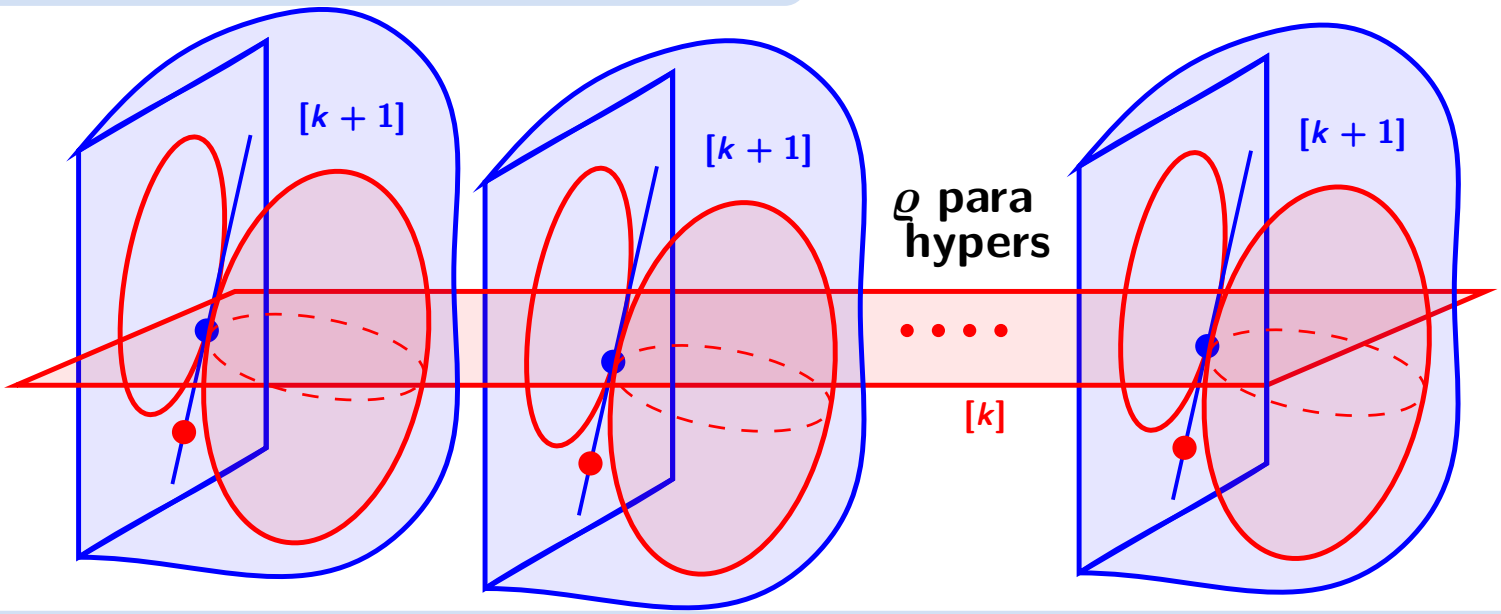
Empty results from the past: a flashback

An inductive fiasco

Corollary

$s_q\Big(k(\varrho+1)+\varrho,\varrho\Big) \leqslant (\varrho+1)\theta_k.$

$\mathbf{PG}\Big(\boldsymbol{k(\varrho+1)+\varrho, q}\Big)$



Theorem (LD, 2019)

Let $k \in \mathbb{N}$, $(k,q) \neq (1,3)$, $q \neq 2$ if k is even and $\varrho < \theta_k$. Then

$$s_q\Big(k(\varrho+1)+\varrho,\varrho\Big) \leqslant (\varrho+1)\theta_k - \varrho.$$

Empty results from the past: a flashback

An inductive fiasco

Theorem (LD, 2019)

Let $k \in \mathbb{N}$, $(k,q) \neq (1,3)$, $q \neq 2$ if k is even and $\varrho < \theta_k$. Then

$$s_q\Big(k(\varrho+1)+\varrho, \varrho\Big) \leqslant (\varrho+1)\theta_k - \varrho.$$

Keep in mind

$$s_q(n,\varrho) \quad \gtrsim \quad \boldsymbol{\varrho \cdot q^{\frac{n-\varrho}{\varrho+1}}}.$$

~~Hypothesis (LD, 2019)~~

Desperate wish (LD, 2020)

$$s_q(n,\varrho) \quad \lesssim \quad \boldsymbol{\varrho \cdot q^{\frac{n-\varrho}{\varrho+1}}},$$

for all n, $\varrho \leqslant n$ and ∞-many q.

So... whatya think!?

Took some work! 😅

Very nice construction. 🙂

Your upper bound doesn't improve ours, though.

... say what now?

Empty results from the past: a flashback

An inductive fiasco

Theorem (LD, 2019)
Let $k \in \mathbb{N}$, $(k, q) \neq (1, 3)$, $q \neq 2$ if k is even and $\varrho < \theta_k$. Then

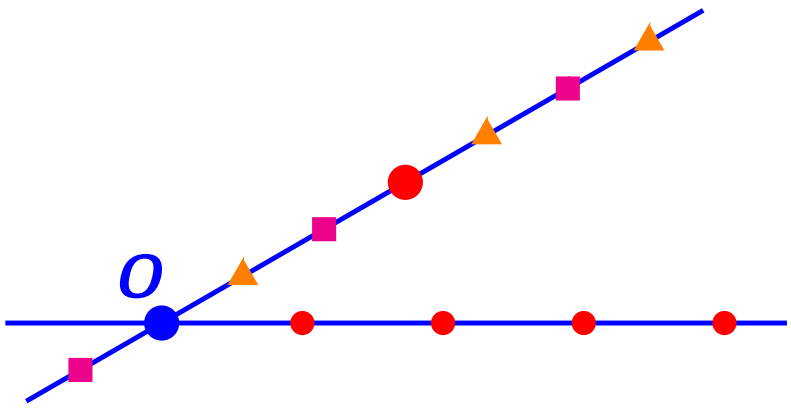
$$s_q\Big(k(\varrho+1)+\varrho, \varrho\Big) \leqslant (\varrho+1)\theta_k - \varrho.$$

Theorem (Davydov, Marcugini & Pambianco, 2019 [6])
Let $k \in \mathbb{N}$ be *large enough*, $q \geqslant 7$ and $\varrho \geqslant 3$. Then

$$s_q\Big(k(\varrho+1)+\varrho, \varrho\Big) \leqslant (\varrho+1)q^k + 2q^{k-1} + \mathcal{O}(q^{k-2}).$$

Empty results from the past: a flashback

- Let $n = 2$ and $\varrho = 1$.
- Let q be square.



Keep in mind

$$s_q(2,1) \gtrsim \sqrt{q}.$$

- 1-sat. set of $\mathrm{PG}(2,q)$.
- Davydov: algebraic proof.
- Combinatorial proof?
  - $\sqrt{q}(\sqrt{q}+1)$ collinear Baer sublines through O.

$$\sqrt{q}(\sqrt{q}+1) = q+q-\sqrt{q}(\sqrt{q}-1).$$

Theorem (Davydov, 1995 [3])
Let q be square. Then

$$s_q(2,1) \leqslant 3\sqrt{q} - 1.$$

Theorem (Davydov et al., 2011 [5])
Let q be a fourth power. Then

$$s_q(2,1) \leqslant 2\sqrt{q} + 2\sqrt[4]{q} + 2.$$

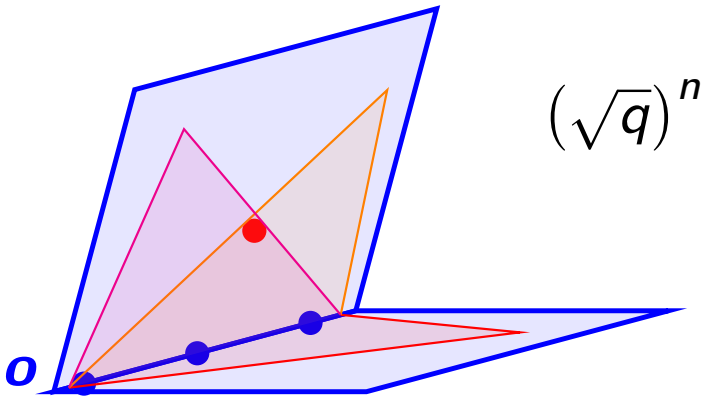
Empty results from the past: a flashback

A better but bitter start

- Let $n \in \mathbb{N}^{\times}$ and $\varrho = 1$.
- Let q be square.

Keep in mind

$$s_q(n,1) \quad \gtrsim \quad q^{\frac{n-1}{2}}.$$



$$(\sqrt{q})^n(\sqrt{q}+1) = 2(\sqrt{q})^{n+1} - (\sqrt{q})^n(\sqrt{q}-1).$$

Two extra obstacles:

1. What if P in 'plane'?
2. What if P in o?

Theorem (LD, 2019)

Let q be square. Then

$$s_q(n,1) \leqslant \begin{cases} 3q^{\frac{n-1}{2}} + 3q^{\frac{n-3}{2}} + \cdots + 3q^{\frac{1}{2}} - \frac{n}{2} & \text{if } n \text{ is even,} \\ 3q^{\frac{n-1}{2}} + 3q^{\frac{n-3}{2}} + \cdots + 3 - \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Empty results from the past: a flashback

A better but bitter start

Theorem (LD, 2019)
Let n be even and q be square. Then

$$s_q(n,1) \leqslant 3q^{\frac{n-1}{2}} + 3q^{\frac{n-3}{2}} + \cdots + 3q^{\frac{1}{2}} - \frac{n}{2}.$$

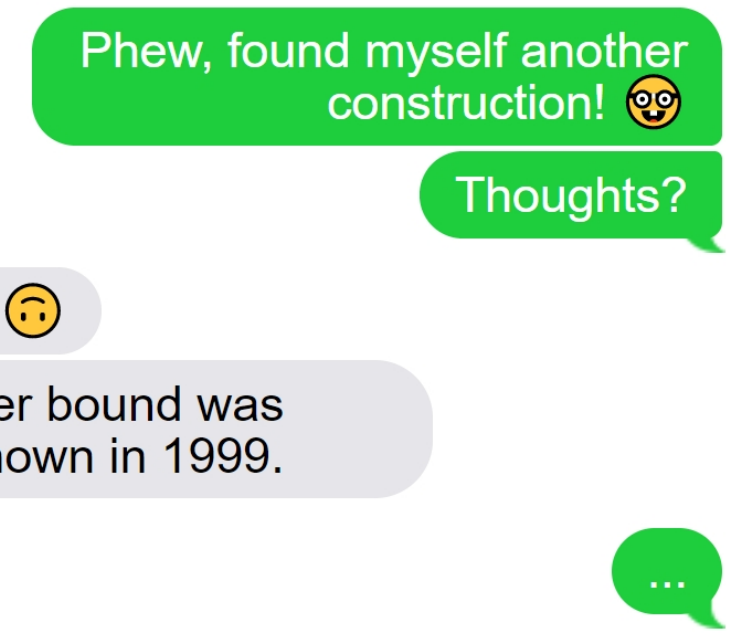
Keep in mind

$$s_q(n,1) \quad \gtrsim \quad q^{\frac{n-1}{2}}.$$

- Davydov et al., 2011 [5]

$$s_q(n,\varrho) \lesssim \binom{n+1}{\varrho} q^{\frac{n-\varrho}{\varrho+1}}$$

if q is a $(\varrho+1)^{th}$ power.



Empty results from the past: a flashback

A better but bitter start

Theorem (LD, 2019)

Let n be even and q be square. Then

$$s_q(n,1) \leqslant 3q^{\frac{n-1}{2}} + 3q^{\frac{n-3}{2}} + \cdots + 3q^{\frac{1}{2}} - \frac{n}{2}.$$

Theorem (Davydov, 1999 [4])

Let n be even and $q \geqslant 16$ be square. Then

$$s_q(n,1) \leqslant 3q^{\frac{n-1}{2}} - q^{\frac{n-2}{2}} + \left\lfloor q^{\frac{n-4}{2}} \right\rfloor.$$

The mixed subgeometry approach

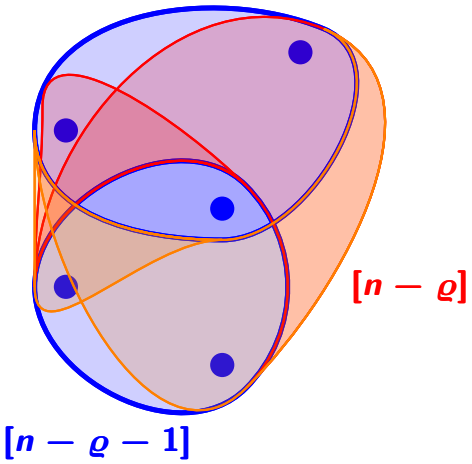
Theorem (Davydov et al., 2011 [5])

Any $(\varrho+1)$-fold strong blocking set in $\mathrm{PG}(n, \sqrt[\varrho+1]{q})$ is a ϱ-saturating set of $\mathrm{PG}(n,q)$.

A **$(\varrho+1)$-fold strong blocking set $\mathcal{B}$** of $\mathrm{PG}(n,q)$:

- Any ϱ-space is spanned by $\varrho+1$ points of $\mathcal{B}$.

How did they obtain the coefficient $\binom{n+1}{\varrho}$?



- Choose $n+1$ independent points.
- Consider span of each $(n-\varrho)$-subset.
- Add the $\varrho+1$ 'extensions' to saturating set $\mathcal{S}$.

$$|\mathcal{S}| \leqslant \binom{n+1}{\varrho+1}(\varrho+1)q^{\frac{n-\varrho}{\varrho+1}}.$$

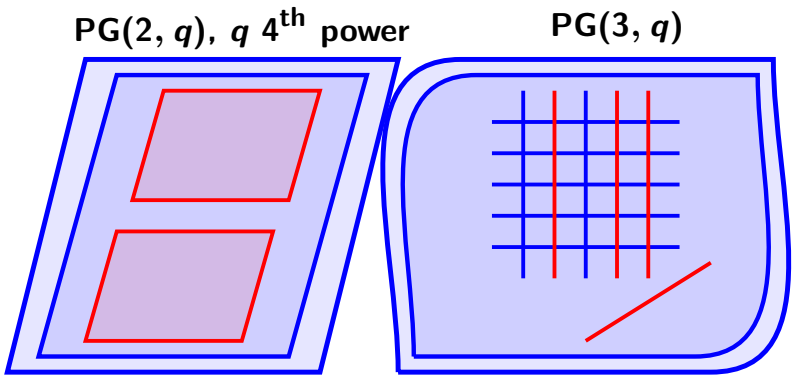
The mixed subgeometry approach

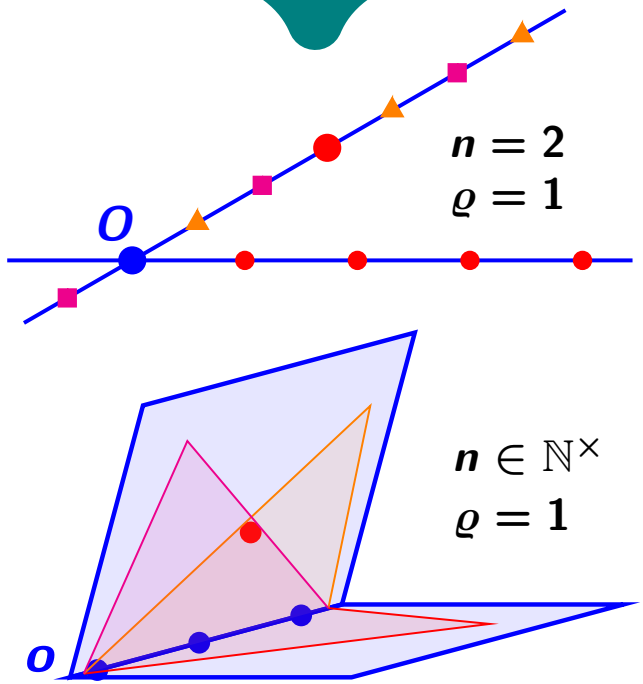
If q is a $(\varrho+1)^{\text{th}}$ power: two possible paths to take

Path of the single subgeometry
Strong blocking set approach

Path of the mixed subgeometries
Mixed subgeometry approach

- $(\varrho+1)$-fold strong blocking sets in $\mathrm{PG}\left(n, \sqrt[\varrho+1]{q}\right)$.

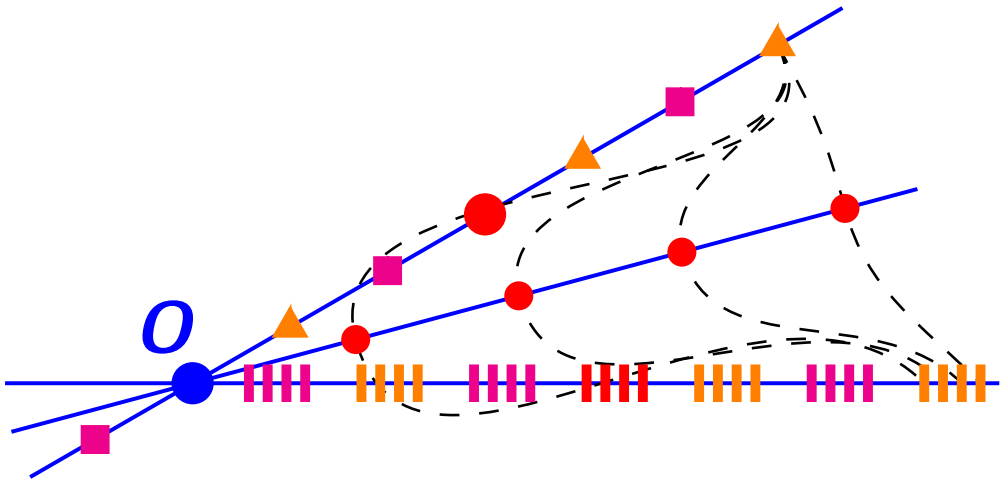


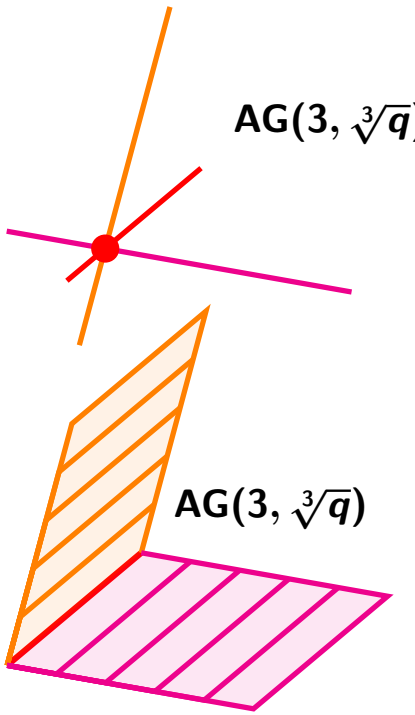


The mixed subgeometry approach

The spark

Let q be cube ($\varrho = 2$).





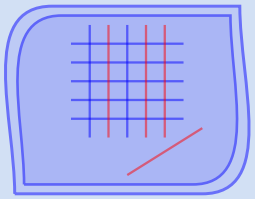
"This last construction looks promising!" - LD, 2019

Theorem (LD, 2019)
Let q be cube. Then

$$s_q(3,2) \leqslant 6\sqrt[3]{q} - 3.$$

Theorem (Davydov et al., 2011 [5])
Let q be cube. Then

$$s_q(3,2) \leqslant 4\sqrt[3]{q} + 4.$$



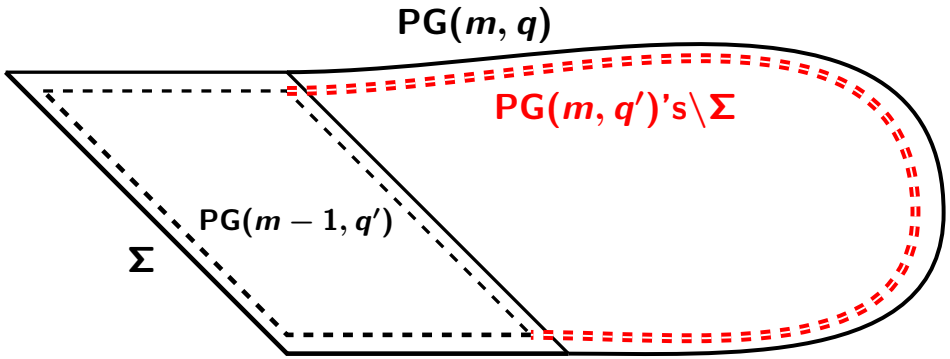
Subgeometries are affine lines

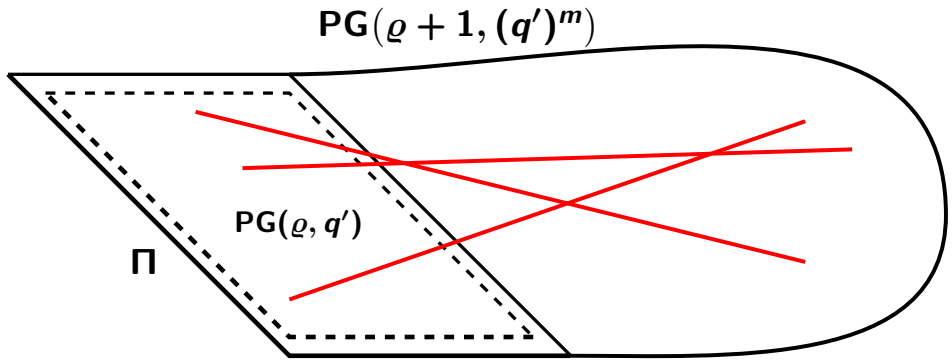
Two isomorphic point-line geometries

Suppose $q = (q')^{\varrho+1}$.

Let $m \in \mathbb{N}^{\times}$, Σ hyperplane of $\mathrm{PG}(m, q)$. $\boldsymbol{Y(\varrho, m, q')} := (\mathcal{P}_{\mathrm{sub}}, \mathcal{L}_{\mathrm{sub}})$.

- $\mathcal{P}_{\mathrm{sub}} := \mathrm{PG}(m, q) \setminus \Sigma$.
- $\mathcal{L}_{\mathrm{sub}} :=$ all $\mathrm{PG}(m, q')$'s$\setminus\Sigma$ through fixed $\mathrm{PG}(m-1, q') \subseteq \Sigma$.





Let $m \in \mathbb{N}^{\times}$, Π hyperpl. of $\mathrm{PG}(\varrho+1, (q')^m)$. $\boldsymbol{T^*(\mathrm{PG}(\varrho, q'))} := (\mathcal{P}_{\mathrm{aff}}, \mathcal{L}_{\mathrm{aff}})$.

- $\mathcal{P}_{\mathrm{aff}} := \mathrm{PG}(\varrho+1, (q')^m) \setminus \Pi$.
- $\mathcal{L}_{\mathrm{aff}} :=$ all affine parts of lines outside of Π intersecting $\mathrm{PG}(\varrho, q') \subseteq \Pi$.

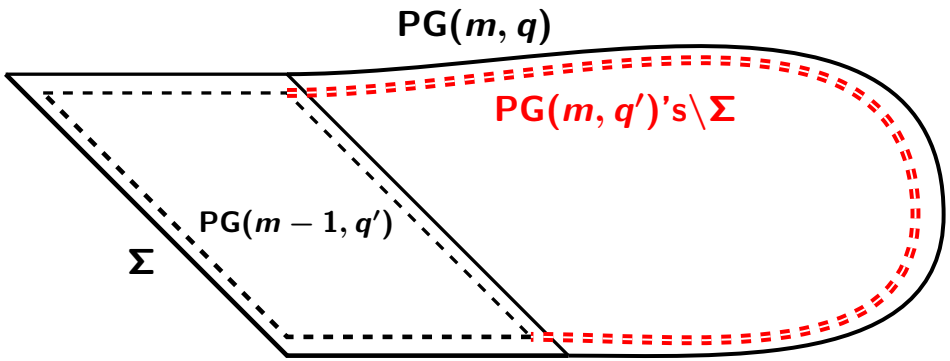
Subgeometries are affine lines

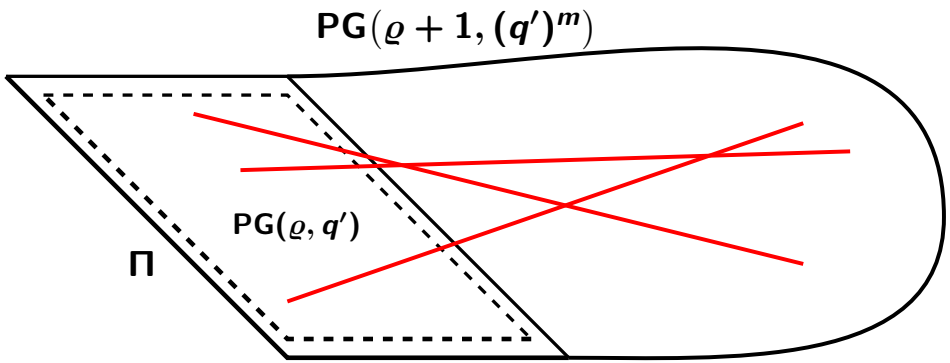
Two isomorphic point-line geometries

Suppose $q = (q')^{\varrho+1}$.

Theorem (LD, 2020 [9])

$$Y(\varrho, m, q') \cong T^*(\mathrm{PG}(\varrho, q')).$$





Theorem (LD, 2020 [9])

Subgeometries are affine lines, really.

Subgeometries are affine lines

Suppose $q = (q')^{\varrho+1}$. Two isomorphic point-line geometries

Theorem (LD, 2020 [9])

$$Y(\varrho, m, q') \cong T^*(\mathrm{PG}(\varrho, q')).$$

Proof.

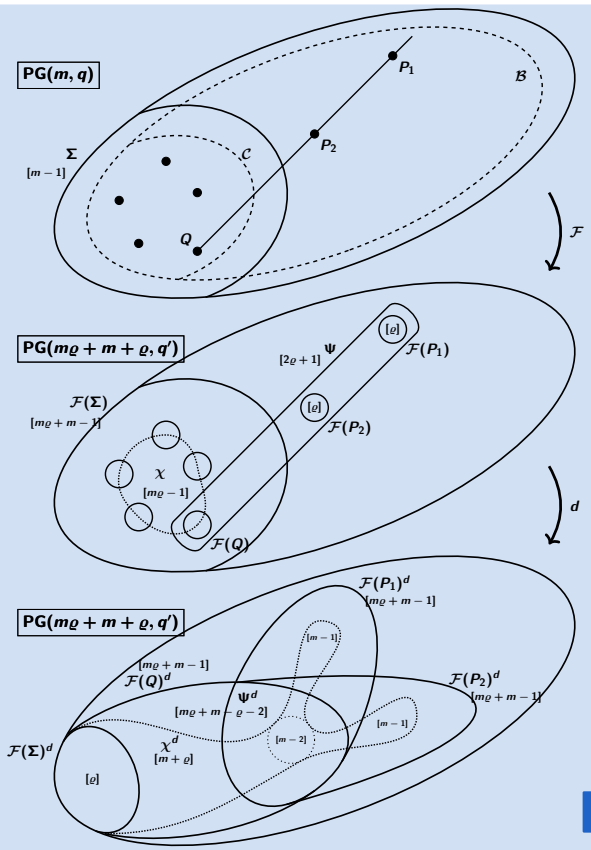
- **Coordinates.**
- Field reduction.

Theorem (LD, 2020 [9])

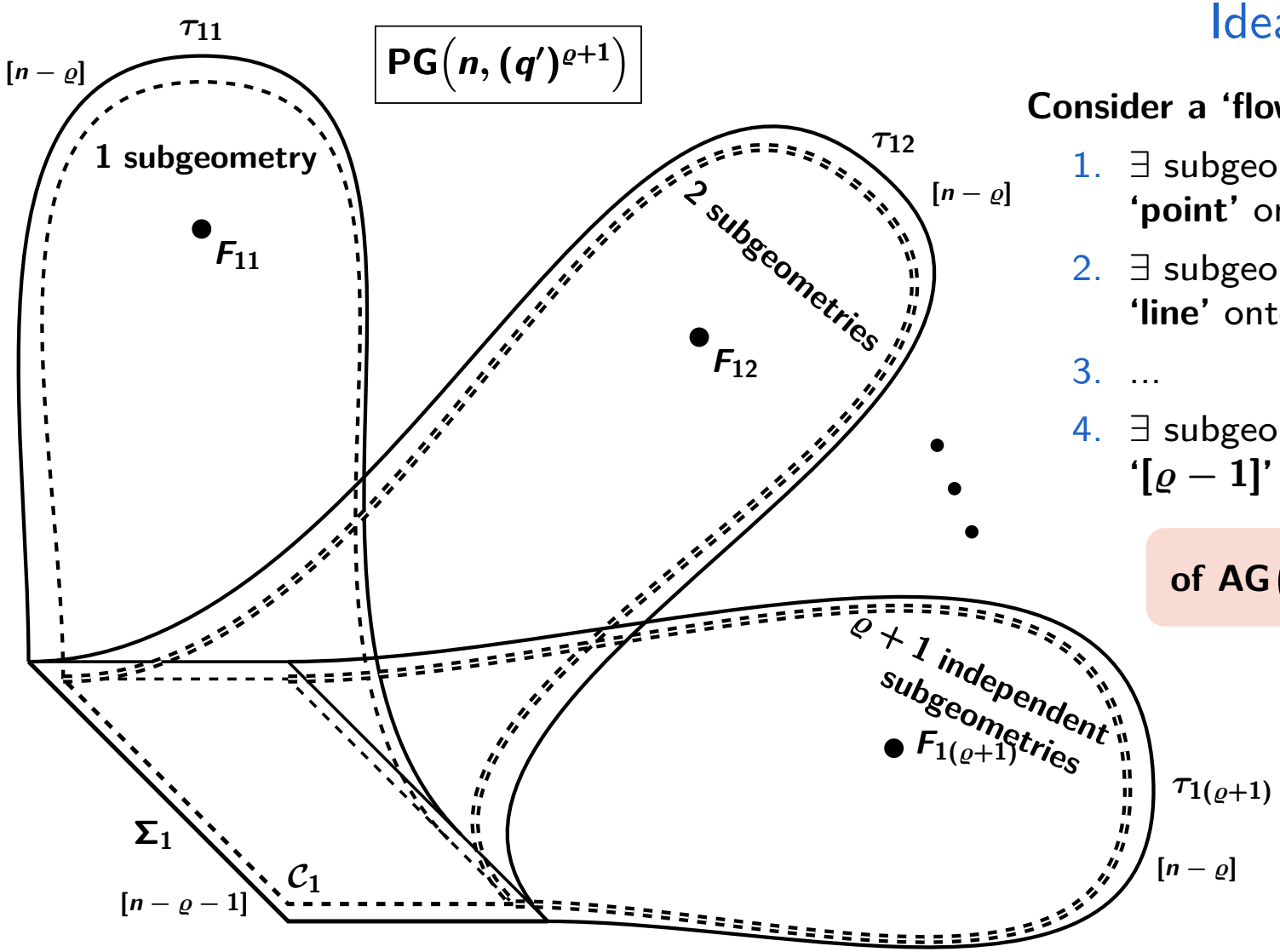
$$Y(\varrho, m, q') \cong X(\varrho, m, q').$$

Theorem (De Winter, Rottey & Van de Voorde, 2015 [8])

$$X(\varrho, m, q') \cong T^*(\mathrm{PG}(\varrho, q')).$$



A construction for general n and ϱ



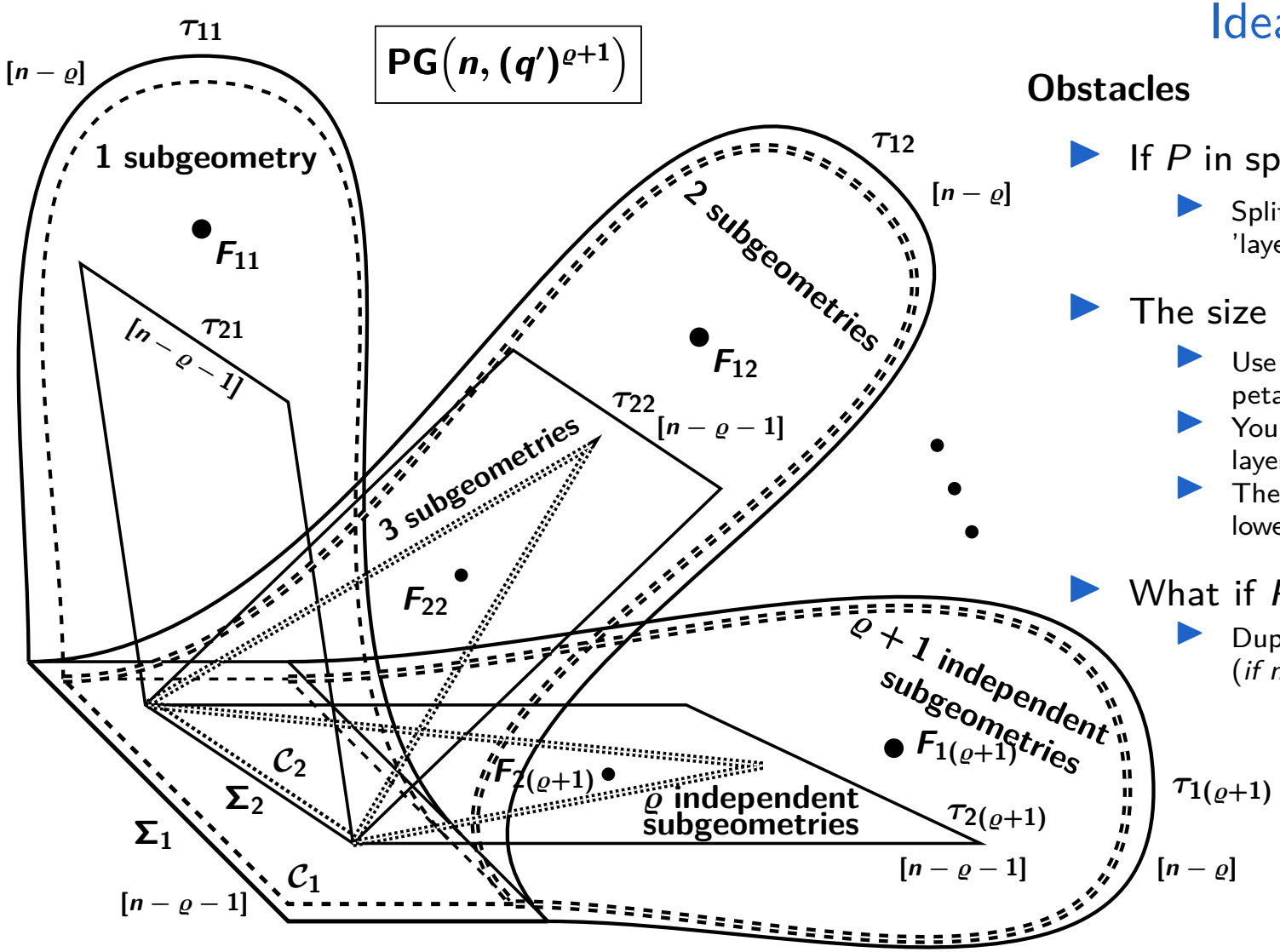
Ideas and obstacles

Consider a 'flower'.

1. $\exists$ subgeom. in τ_{11} that projects **'point'** onto **'line'**,
2. $\exists$ subgeom. in τ_{12} that projects **'line'** onto **'plane'**,
3. ...
4. $\exists$ subgeom. in $\tau_{1\varrho}$ that projects **'$[\varrho-1]$'** onto **'hyperplane'**

of $\mathrm{AG}\left(\varrho+1,(q')^{n-\varrho}\right)$.

A construction for general n and ϱ

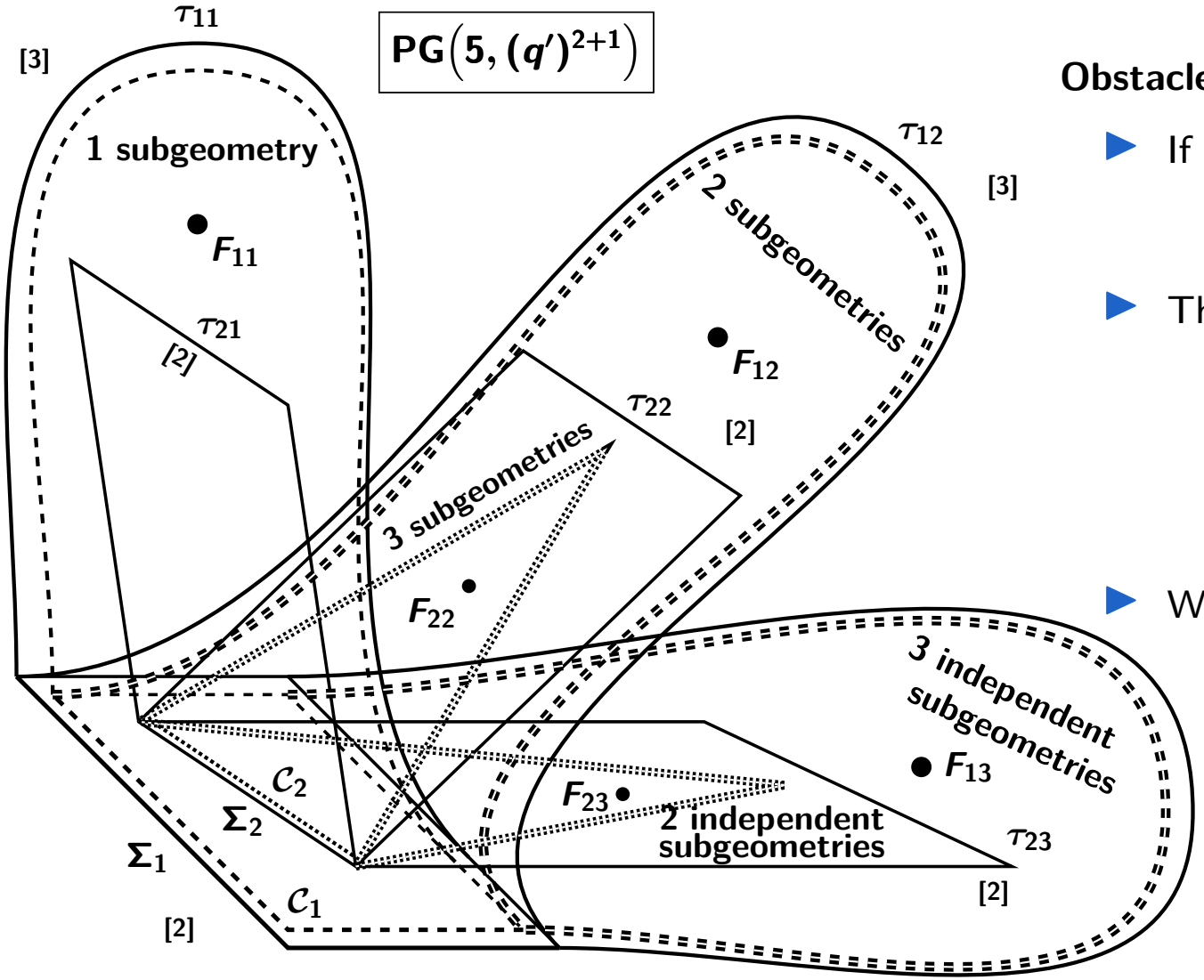


Ideas and obstacles

Obstacles

- If P in span of $< (\varrho+1)$ petals?
  - Split petals and add multiple 'layers' in each petal!
- The size will get expon. **big**...
  - Use the subgeometries from the petal above!
  - You only need $\min\{\varrho, n-\varrho\}$ layers, and not in *all* petals!
  - The number of subgeometries in lower layers can be reduced!
- What if $P \in \Sigma_1$?
  - Duplicate construction! (*if necessary*)

A construction for general n and ϱ



Ideas and obstacles

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5 A construction for general n and ϱ

One bound to rule them all

Theorem (LD, 2020 [9])

Let $0 < \varrho < n$ and let $q = (q')^{\varrho+1}$ for any prime power q'. Then

$$s_q(n,\varrho) \leqslant \sum_{i=1}^{k(n,\varrho)} \left(\frac{(\varrho+1)(\varrho+2)}{2} (q')^{n+1-i(\varrho+1)} \right) + \sum_{i=1}^{k(n,\varrho)-1} \sum_{j=1}^{\varrho-1} \tilde{a}(\varrho,j)(q')^{n+1-i(\varrho+1)-j}$$
$$+ \sum_{j=1}^{\ell(n,\varrho)-1} \overline{a}(n,\varrho,j)(q')^{\ell(n,\varrho)-j} - \tilde{c}(n,\varrho) - \overline{c}(n,\varrho)$$
$$+ \delta_{q'=2} \cdot \left((2^{\varrho-1}-1) \cdot \sum_{i=1}^{k(n,\varrho)-1} \left(2^{n-\varrho+2-i(\varrho+1)} \right) + 2^{\ell(n,\varrho)} - 2 \right),$$

with

- $k(n,\varrho) := \left\lceil \frac{n-\varrho}{\varrho+1} \right\rceil$,
- $\ell(n,\varrho) := \left(n \pmod{\varrho+1} \right) + 1$,
- $\tilde{a}(\varrho,j) := \frac{\varrho(\varrho+2j+1)-j(3j+1)}{2}$,
- $\overline{a}(n,\varrho,j) := \frac{\ell(n,\varrho)\left(2\varrho-\ell(n,\varrho)+2j+1\right)-j(3j+1)}{2}$,
- $\tilde{c}(n,\varrho) := \left(k(n,\varrho)-1\right)\frac{\varrho^2(\varrho+1)}{2}$,
- $\overline{c}(n,\varrho) := \frac{\varrho(\varrho+1)+\ell(n,\varrho)\left(\ell(n,\varrho)-1\right)\left(2\varrho-\ell(n,\varrho)+1\right)}{2}$,
- $\delta_{q'=2} := \begin{cases} 1 & \text{if } q' = 2, \\ 0 & \text{if } q' \neq 2. \end{cases}$

A construction for general n and ϱ

One bound to rule them all

Corollary (LD, 2020 [9])
Let $1 < \varrho < n$ and let $q = (q')^{\varrho+1}$ for any prime power q'. Then

$$s_q(n,\varrho) \leqslant \frac{(\varrho+1)(\varrho+2)}{2}(q')^{n-\varrho} + \varrho(\varrho+1)\left((q')^{n-\varrho-1} + \cdots + q' + 1\right).$$

Keep in mind

$$s_q(n,\varrho) \quad \gtrsim \quad \varrho \cdot q^{\frac{n-\varrho}{\varrho+1}}.$$

~~**Hypothesis (LD, 2019)**~~
Desperate wish (LD, 2020)

$$s_q(n,\varrho) \quad \lesssim \quad \varrho \cdot q^{\frac{n-\varrho}{\varrho+1}},$$

for all n, $\varrho \leqslant n$ and ∞-many q.

Davydov et al., 2011 [5]

$$s_q(n,\varrho) \lesssim \binom{n+1}{\varrho} q^{\frac{n-\varrho}{\varrho+1}}$$

if q is a $(\varrho+1)^{th}$ power.

A construction for general n and ϱ

One bound to rule them all

Corollary (LD, 2020 [9])
Let $1 < \varrho < n$ and let $q = (q')^{\varrho+1}$ for any prime power q'. Then

$$s_q(n,\varrho) \leqslant \frac{(\varrho+1)(\varrho+2)}{2}(q')^{n-\varrho} + \varrho(\varrho+1)\left((q')^{n-\varrho-1} + \cdots + q' + 1\right).$$

Coincidental example found in the wild:

Theorem (Davydov et al., 2011 [5])
Let $q = (q')^3$ for any prime power q'. Then

$$s_q(4,2) \leqslant 9(q')^2 - 8q' + 4.$$

Corollary (LD, 2020 [9])
Let $q = (q')^3$ for any prime power q'. Then

$$s_q(4,2) \leqslant 6(q')^2 + 3q' - 6 \ (+2 \text{ if } q = 8).$$

A construction for general n and ϱ

arXiv:2008.13459

Slides: **https://users.ugent.be/~ldnaux**

Thank you for listening

Any questions?

Suggestions?

Funny anecdotes?

References

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[2] **D. Bartoli, A. A. Davydov, M. Giulietti, S. Marcugini, and F. Pambianco.** *New bounds for linear codes of covering radii* 2 *and* 3, volume 11. 2019.

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[5] **A. A. Davydov, M. Giulietti, S. Marcugini, and F. Pambianco.** *Linear nonbinary covering codes and saturating sets in projective spaces*, volume 5. 2011.

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[8] **S. De Winter, S. Rottey, and G. Van de Voorde.** Linear representations of subgeometries. *Des. Codes Cryptogr.*, 77(1):203–215, 2015.

[9] **L. Denaux.** Constructing saturating sets in projective spaces using subgeometries. *arXiv:2008.13459*, 2020.