Combinatorics 2022

Higgledy-piggledy sets

in projective spaces

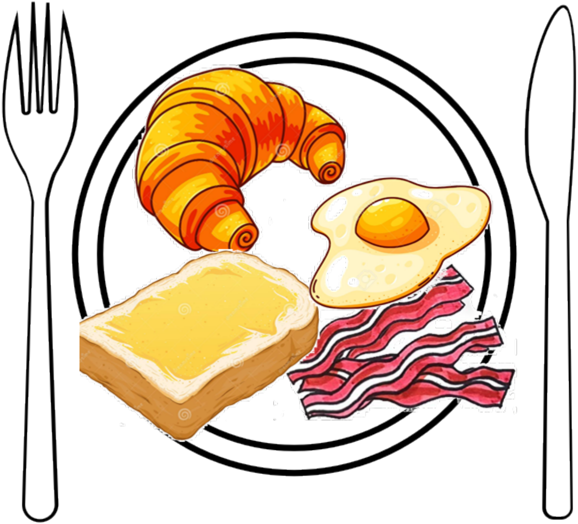
Lins Denaux

2nd of June 2022

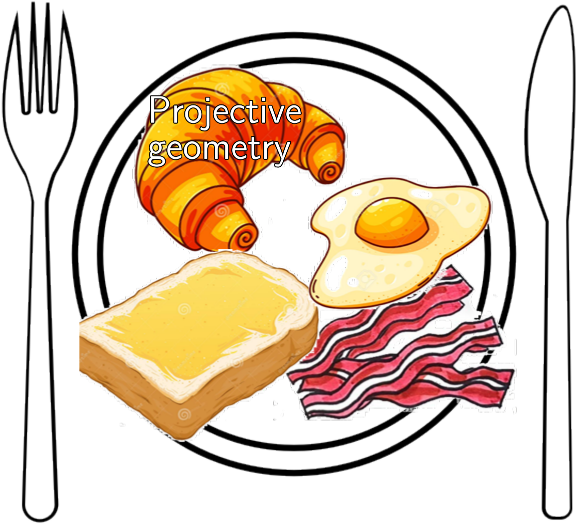
GHENT UNIVERSITY

Overview

On the menu



On the menu



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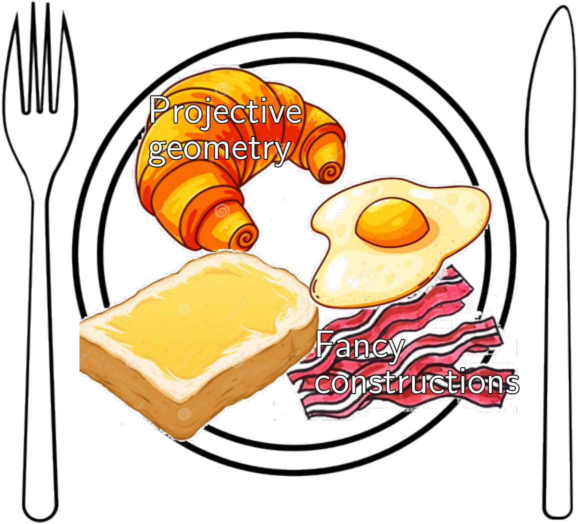

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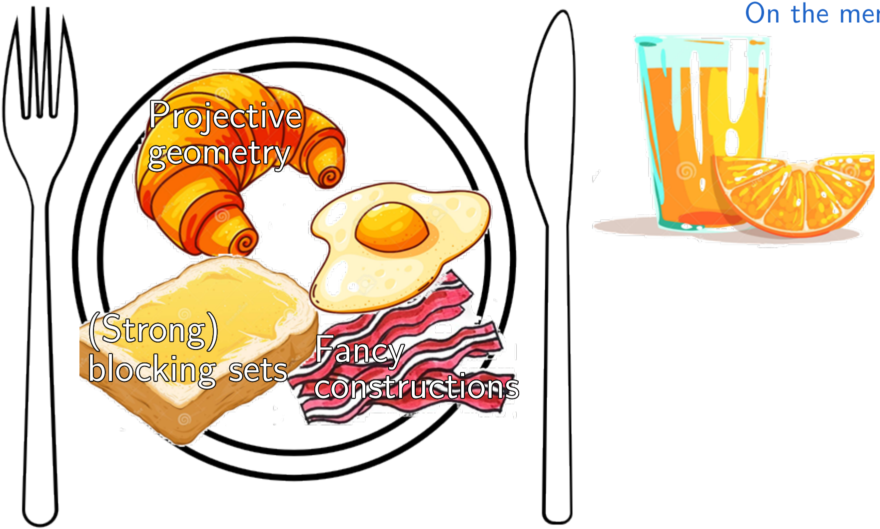
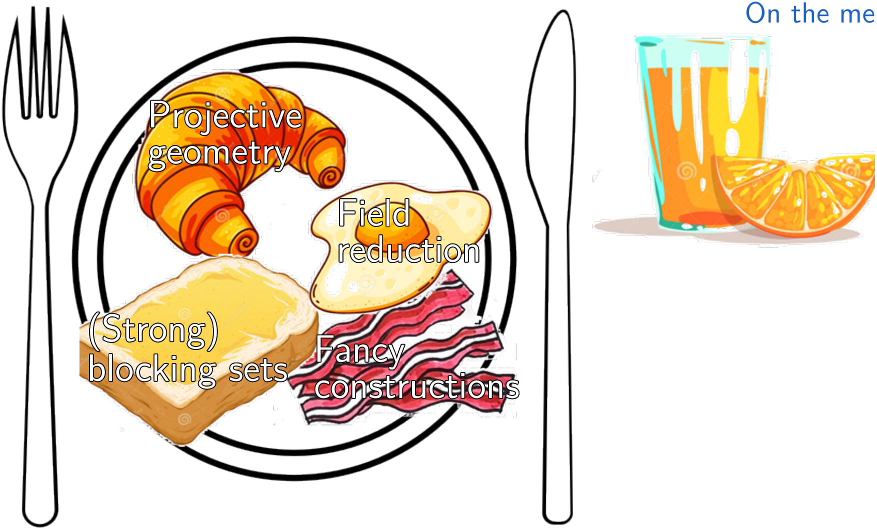

Projective geometry
(Strong) blocking sets
Fancy constructions

On the menu


Projective geometry
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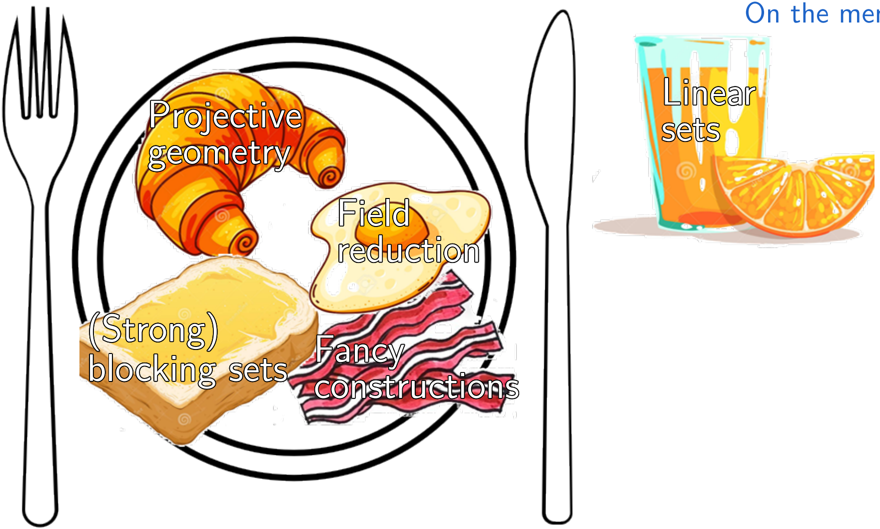
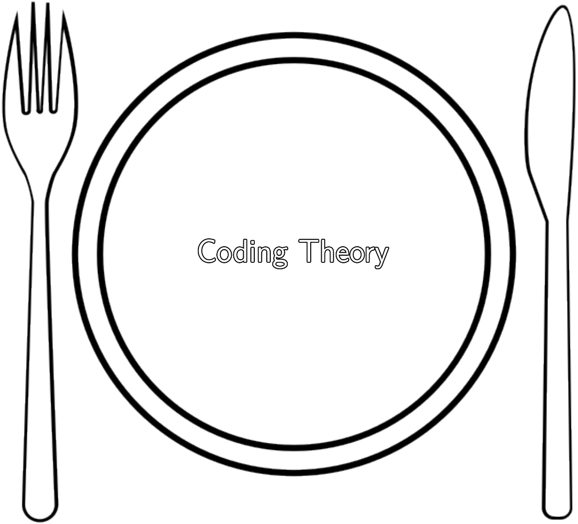


Projective geometry
Field reduction
(Strong) blocking sets
Fancy constructions
Linear sets

1 Introduction

On the menu



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A *$(N-k+1)$-fold k-blocking set* of $\mathrm{PG}(N, q)$ is a point set that meets every $(N-k)$-dimensional space in at least $N-k+1$ points.

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A *strong k-blocking set* of $\mathrm{PG}(N, q)$ is a point set that meets every $(N-k)$-dimensional space κ in a point set spanning κ.

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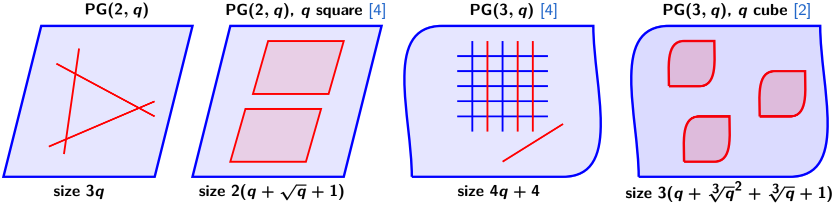

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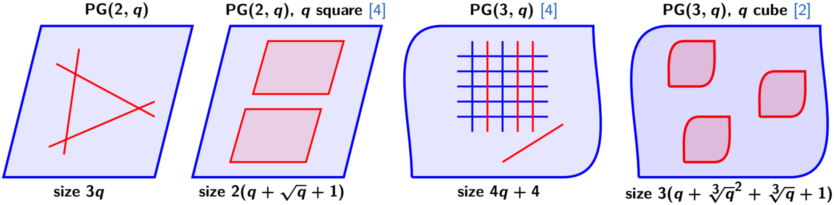

A *higgledy-piggledy set of k-spaces* is a set $\mathcal{K}$ of k-spaces such that the point set $\cup\mathcal{K}$ is a strong k-blocking set.

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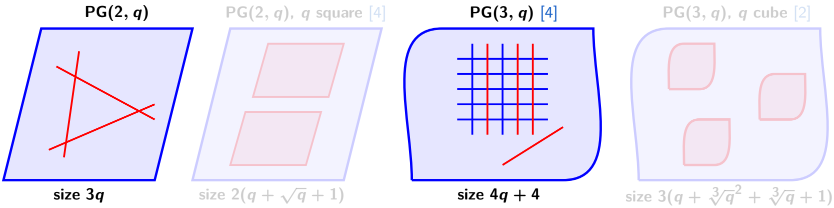

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Motivation

Let $n \in \mathbb{N}^{\times}$ and $k' \in \{0, 1, \ldots, n\}$.

Consider (q-ary) linear $[n, k']_q$-code $\mathcal{C}$ (k'-space of $\mathrm{V}(n, q)$).

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- Elements/vectors of $\mathcal{C}$ are called *codewords*.
- The *support* $\mathrm{supp}(c)$ of a codeword $c \in \mathcal{C}$ is the subset of $\{1, 2, \ldots, n\}$ of non-zero positions in c.

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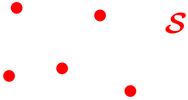
A codeword $c \in \mathcal{C}$ is called *minimal* if

$$\forall c' \in \mathcal{C} : \mathrm{supp}(c') \subseteq \mathrm{supp}(c) \Rightarrow c' \in \langle c \rangle_{\mathbb{F}_q}.$$

A linear code is called *minimal* if all its codewords are minimal.

Motivation

Correspondence with minimal codes



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2 Motivation

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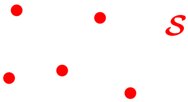

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$$\begin{matrix} P_1 & P_2 & P_3 & & P_i & & P_{|\mathcal{S}|} \end{matrix}$$

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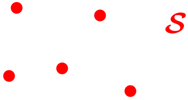

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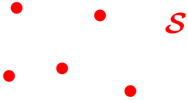

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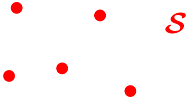

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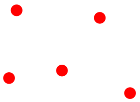
Goal: Finding small strong blocking sets.

Motivation

Correspondence with minimal codes



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Goal: Finding small ~~strong blocking sets~~ higgledy-piggledy sets.

Theorem (Sz. L. Fancsali, P. Sziklai [5]).
If $q \geqslant N + \lfloor \frac{N}{2} \rfloor$, then
A higg.-pigg. line set contains at least $N + \lfloor \frac{N}{2} \rfloor$ lines.

Known results

Theorem (Fancsali, Sziklai [5]; T. Héger, Z. L. Nagy [7]).
If $q \geqslant N + \lfloor \frac{N}{2} \rfloor$, then
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The line case ($k = 1$)

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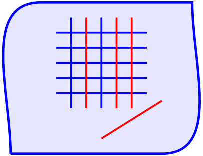
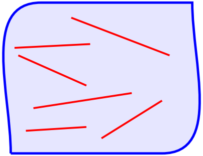


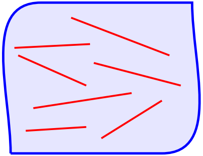
size $4q + 4$

PG(4, q)
$q > 36086$, char(q) $\neq 2, 3$ [3]



size $6q + 6$

PG(5, q) [2]



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eg. consider one line parity case.

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$k \to N-k-1$
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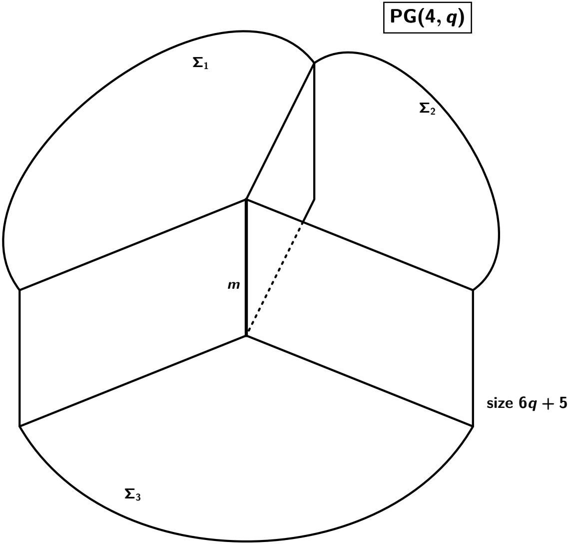
Elem. geometry
intersecting six lines, see next slide.

An example



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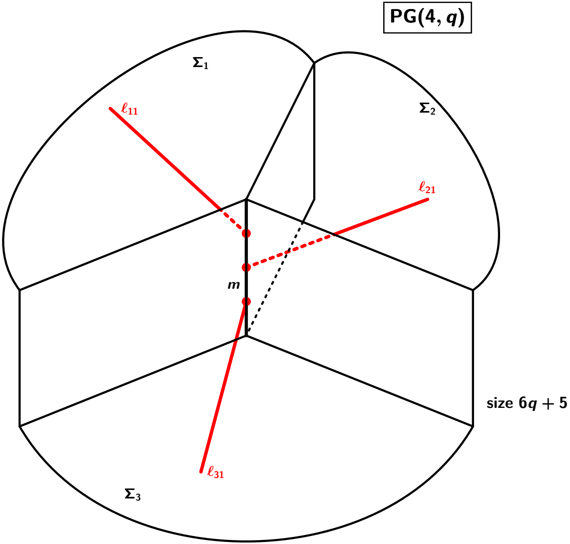

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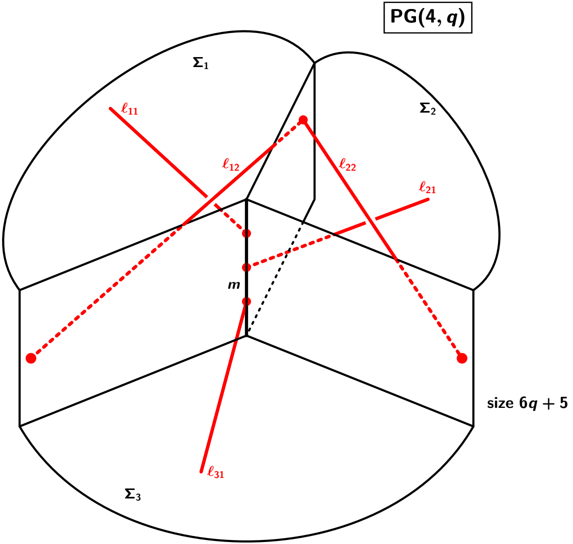

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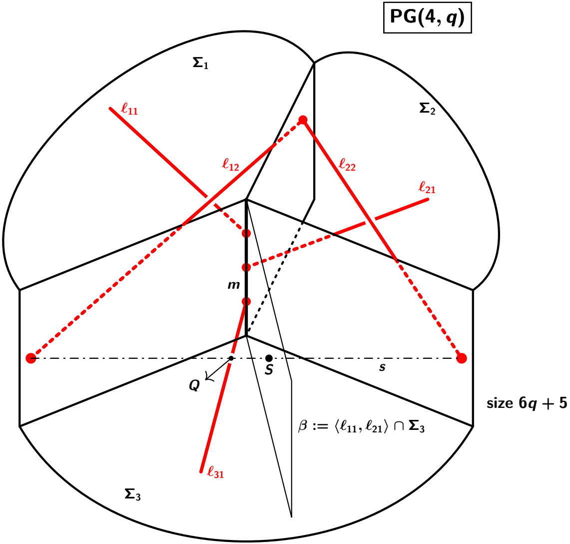

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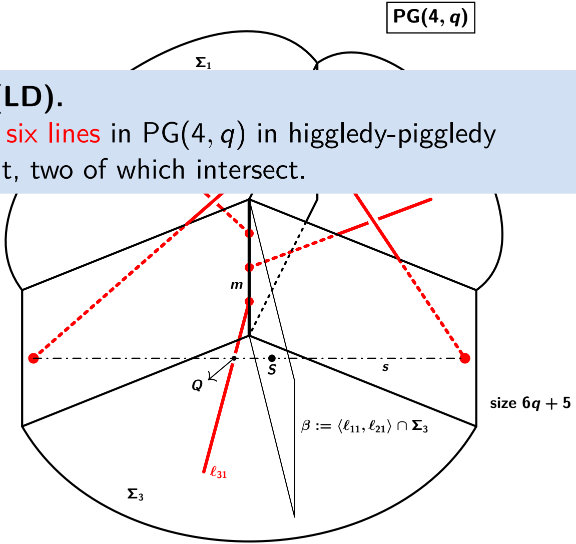
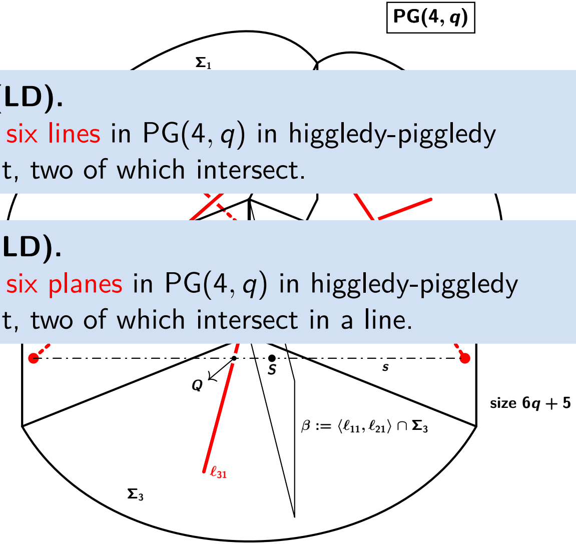


Theorem (LD).
There exist six lines in PG$(4, q)$ in higgledy-piggledy arrangement, two of which intersect.



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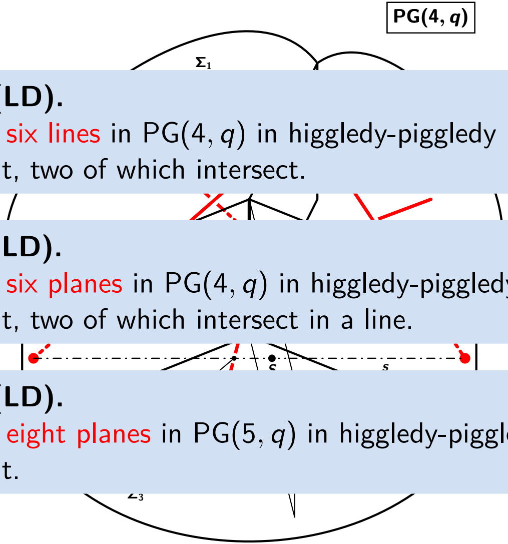

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Theorem (LD).
There exist eight planes in $\mathrm{PG}(5, q)$ in higgledy-piggledy arrangement.

5

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Open problems and possible future approaches

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- Find less than $2N-1$ higg.-pigg. lines, towards $\left\lfloor \frac{3}{2}N \right\rfloor$?

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L. Denaux. Higgledy-piggledy sets in projective spaces of small dimension. *Electron. J. Comb.*, accepted. (arXiv:2109.08572)

Thank you for listening

Any questions?

Suggestions?

Marvellous revelations?

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