Colloquium Coding Theory and Cryptography

Small weight code words

in the code of points and hyperplanes in $PG(n, q)$

Lins Denaux

Joint work with S. Adriaensen, L. Storme and Zs. Weiner

GHENT UNIVERSITY

1 Preliminaries

The code $C_{n-1}(n, q)$

Vector space over $\mathbb{F}_p$ spanned by the rows of the incidence matrix of hyperplanes and points in $PG(n, q)$. Vectors $=$ '**code words**'.

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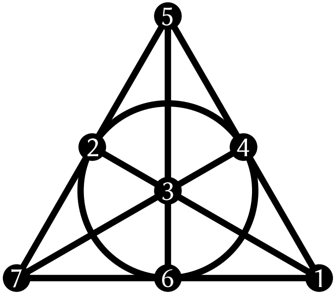

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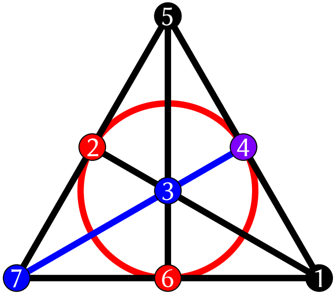

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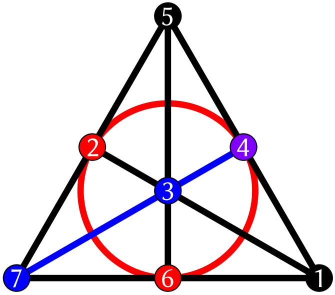
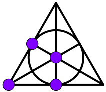


$\color{red}{\text{red}} + \color{blue}{\text{blue}} = (0\,1\,1\,0\,0\,1\,1) =$ 

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$\mathrm{wt}(c) \leqslant \max\{3p + 1, 4p - 22\}$

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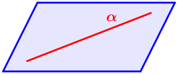

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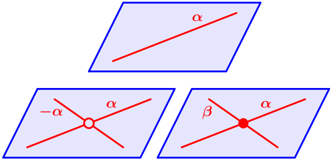

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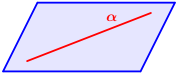

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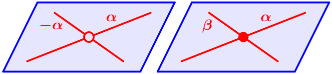

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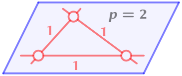

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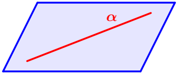

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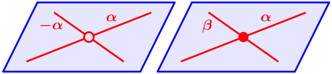

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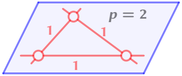

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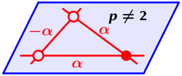

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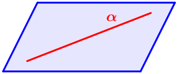

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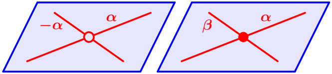

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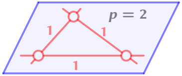

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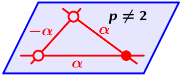

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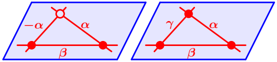

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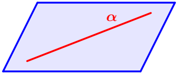

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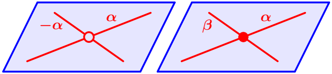

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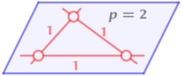

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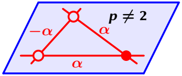

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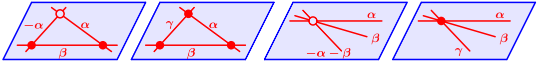

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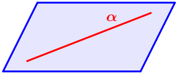

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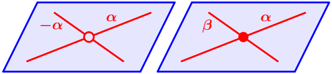

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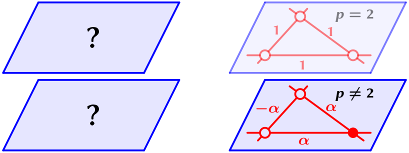

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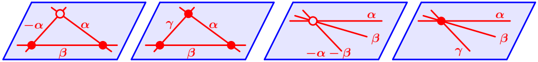

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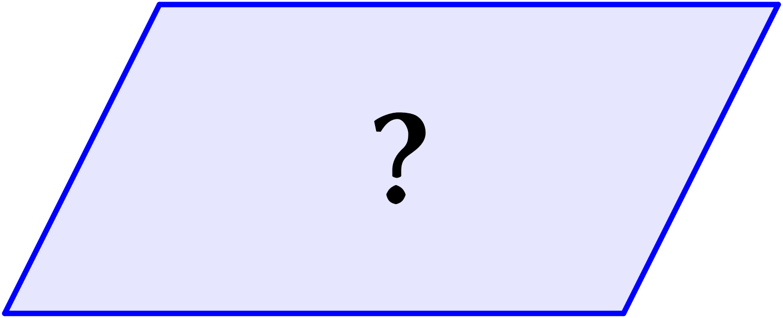

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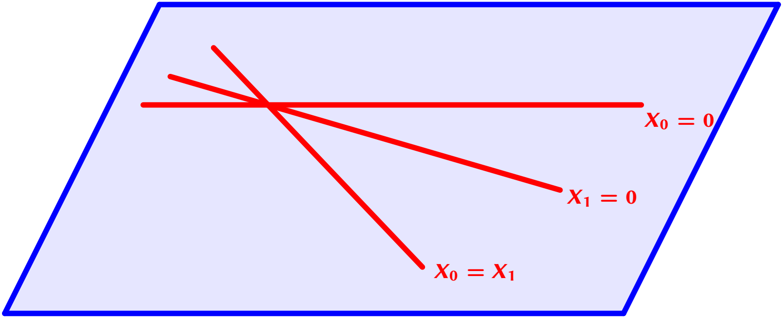

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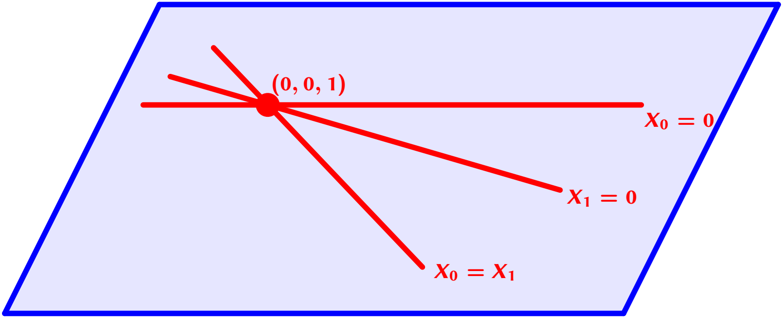

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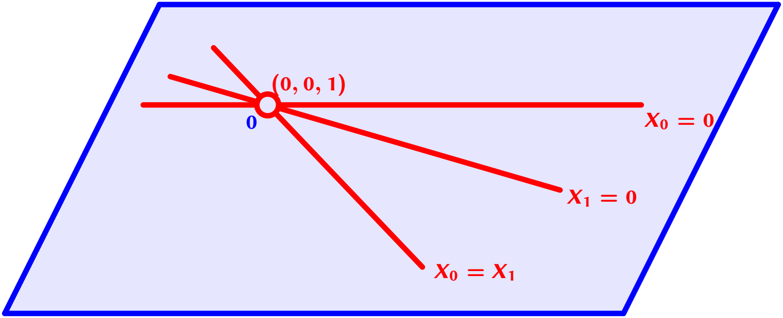

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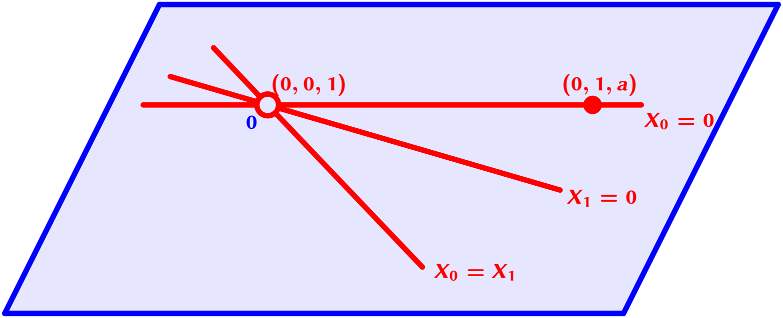

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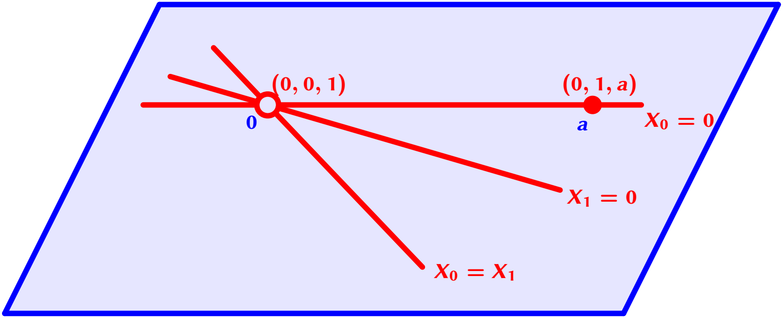

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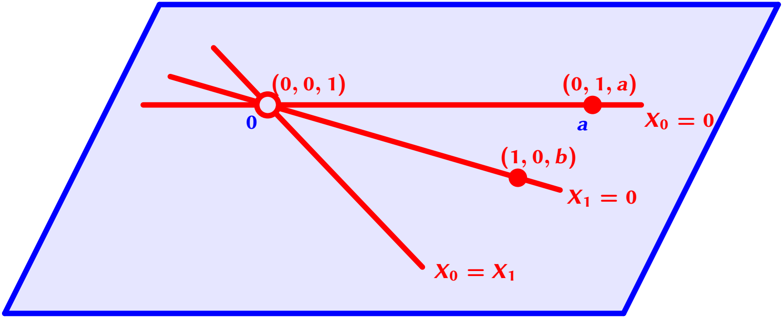

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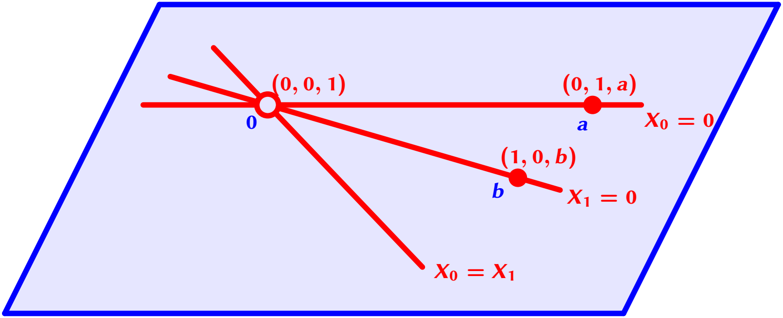

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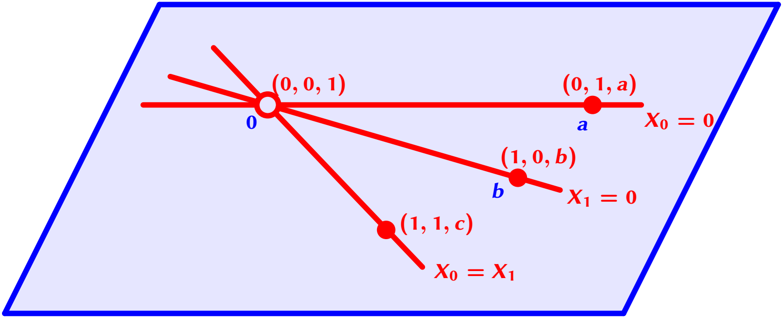

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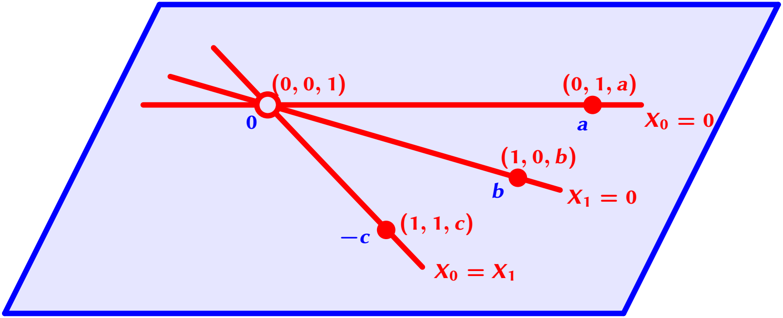

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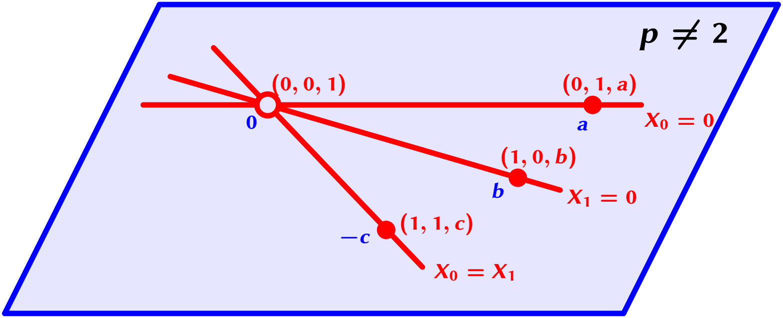

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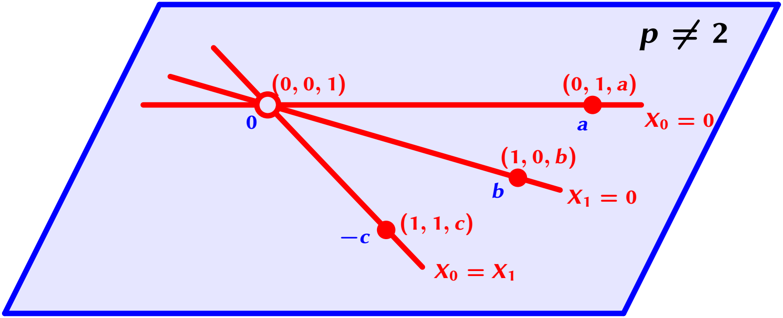

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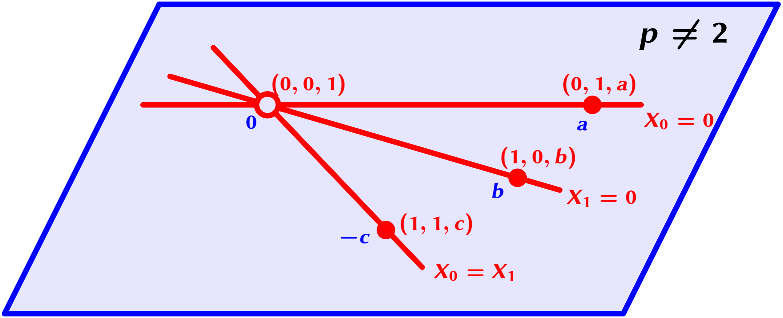

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- $\dim(C_1(2,p^h)) = \binom{p+1}{2}^h + 1$.
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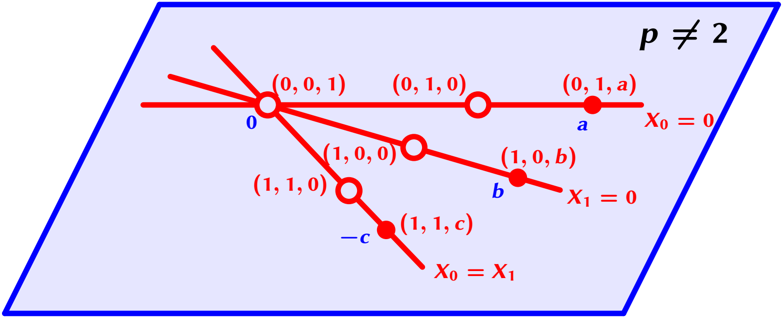

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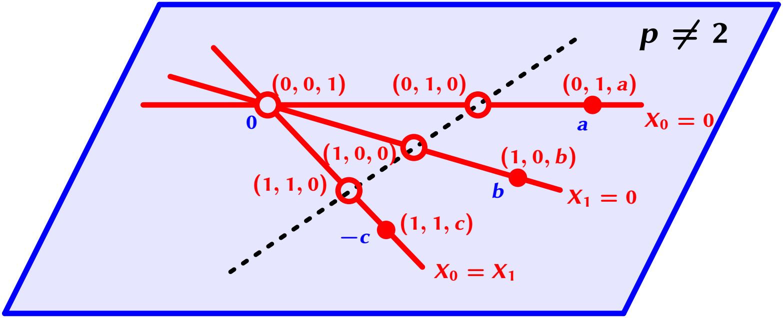

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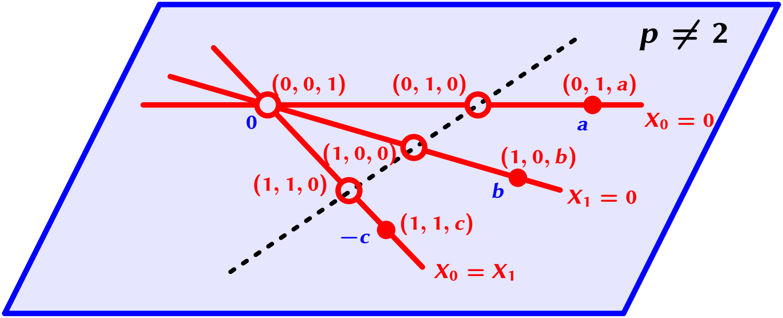

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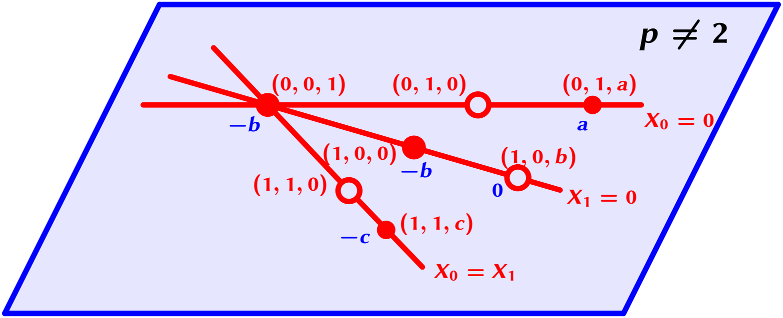

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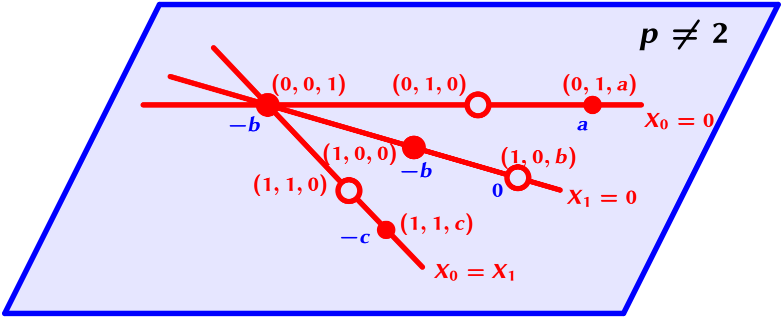

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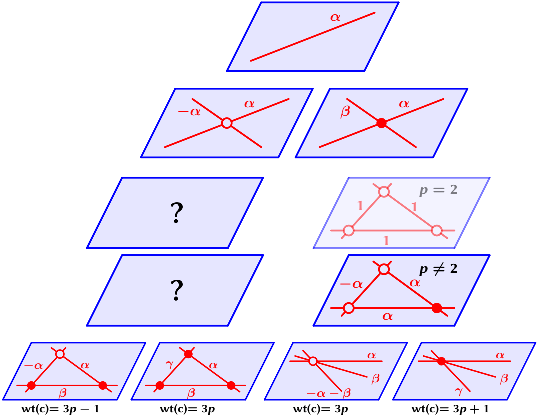

$\mathrm{wt}(c) \leqslant \max\{3p + 1, 4p - 22\}$

2 Known results in the plane: $C_1(2, q)$

Small weight code words $\approx$ **few** hyperplanes (= lines)?

The case $q = p$ prime (Szőnyi & Weiner):

weight

$\mathrm{wt(c)} = p + 1$

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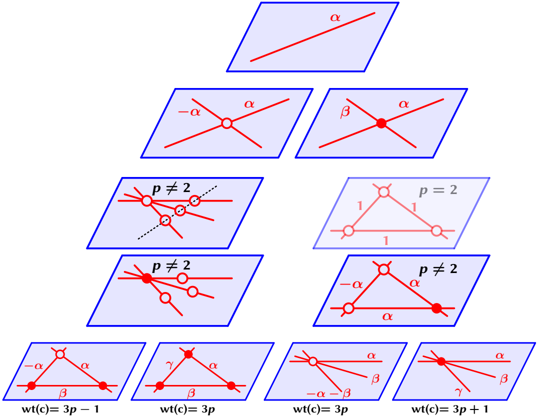

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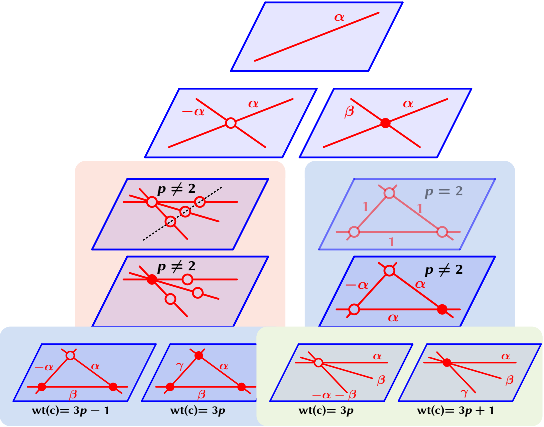

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3 Known results in general: $C_{n-1}(n, q)$

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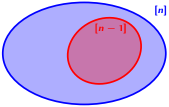
$$\mathbf{wt(c)} = q^{n-1} + \cdots + q + 1$$



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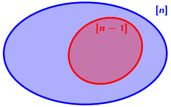

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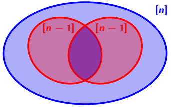

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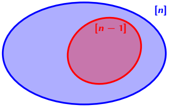

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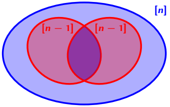

First result: classification of the third smallest weight

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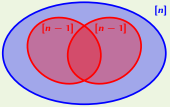

Proof of third smallest weight: first approach

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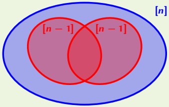

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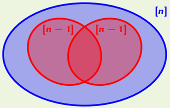

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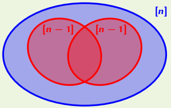

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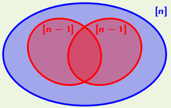

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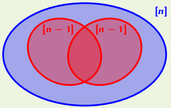

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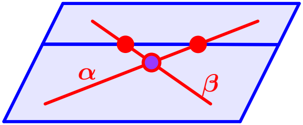

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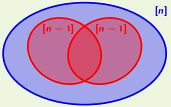

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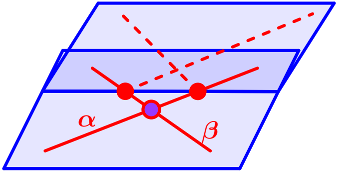

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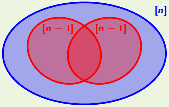

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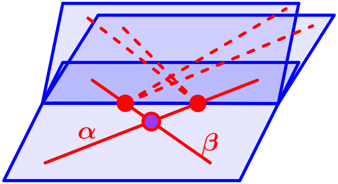

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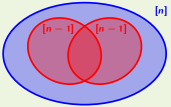

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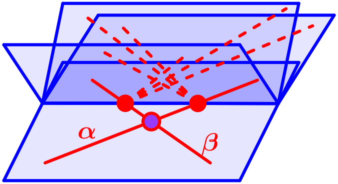

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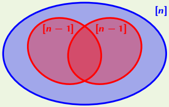

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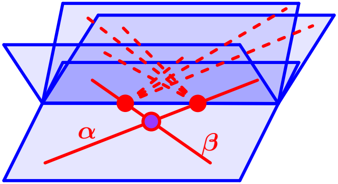

4 Proof of third smallest weight: first approach

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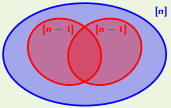
$$\mathrm{wt}(c) = 2q^{n-1} + \cdots + q + 1$$



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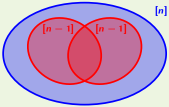

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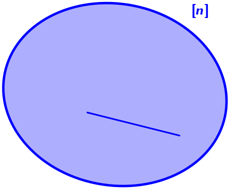

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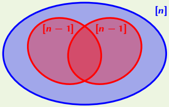

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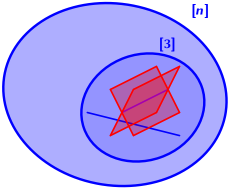

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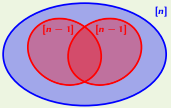

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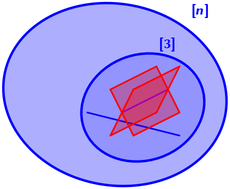

- Non-characterized spaces: contains affine part.
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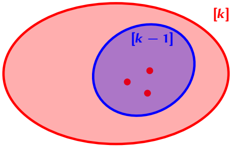

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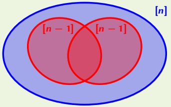

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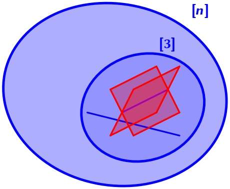

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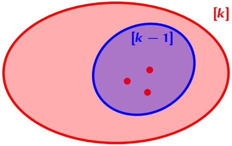
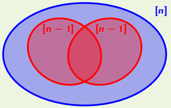


A quiet moment to think things through

First result: classification of the third smallest weight

$$\mathrm{wt}(c) = 2q^{n-1} + \cdots + q + 1$$



for all c with $2q^{n-1} < wt(c) \leqslant \frac{5}{2}q^{n-1} - \frac{7}{5}q^{n-2}$.

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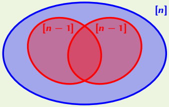
$$\mathbf{wt(c)} = 2q^{n-1} + \cdots + q + 1$$



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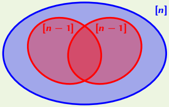

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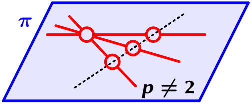
- 'Weird' code word c in plane π (for $q = p$ **prime**).



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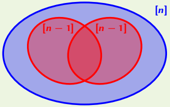

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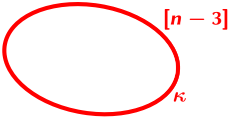
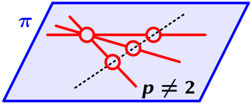
- Chose a disjoint $(n-3)$-space κ.





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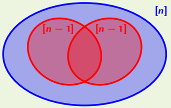
$$\mathbf{wt(c) = 2q^{n-1} + \cdots + q + 1}$$



for all $\boldsymbol{c}$ with $\boldsymbol{2q^{n-1} < wt(c) \leqslant \frac{5}{2}q^{n-1} - \frac{7}{5}q^{n-2}}$.

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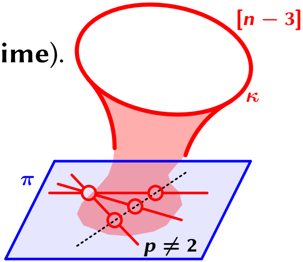

If $c = \sum_i \alpha_i l_i$, then $c' := \sum_i \alpha_i \langle l_i, \kappa \rangle$ is a linear combination of hyperplanes; $\boldsymbol{wt(c') = 3p^{n-1} - 3p^{n-2}}$.

A quiet moment to think things through

Small weight code words $\approx$ **few** hyperplanes (= lines)?

The case $q = p$ prime (Szőnyi & Weiner):

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$\mathrm{wt}(c) = p + 1$

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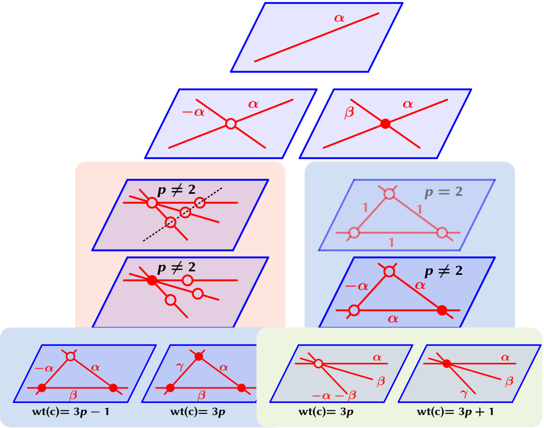

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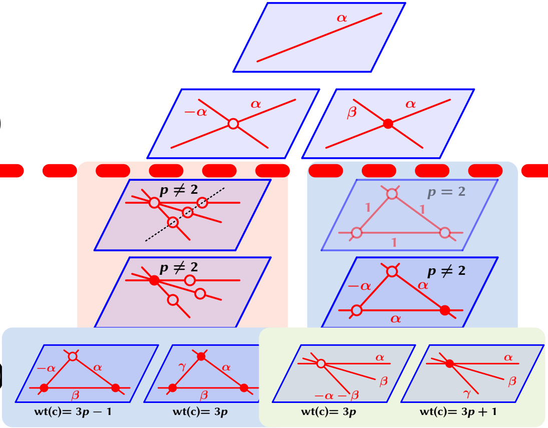

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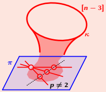
Conjecture ($q = p$ prime, $p \geqslant 7$)

Code words up to weight $3p^{n-1} + p^{n-2} + \cdots + p + 1$
$\sim$ linear combinations of at most three hyperplanes
or a generalization of the ‘weird’ code word right.

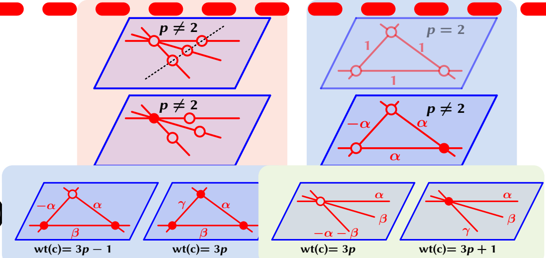


weight

wt(c) $= 3p - 3$

wt(c) $= 3p - 2$

wt(c) $= 3p\,(\pm 1)$



wt(c) $\leqslant \max\{3p + 1, 4p - 22\}$

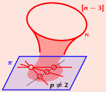
5 A quiet moment to think things through

Small weight code words ≈ **few** hyperplanes (= lines)?

The case $q = p$ prime (Szőnyi & Weiner):

Conjecture ($q = p$ prime, $p \geqslant 7$)

Code words up to weight $3p^{n-1} + p^{n-2} + \cdots + p + 1$
$\sim$ linear combinations of at most three hyperplanes
or a generalization of the 'weird' code word right.



We can likely do better.

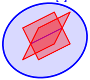
6 The Budapest effect

6 The Budapest effect

Before Budapest

Proof in $C_2(3, q)$.

[3]



6 The Budapest effect

Before Budapest

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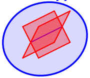
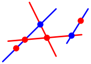


$(0, 1, 2 \parallel q, q + 1)$-secants.



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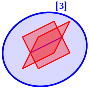

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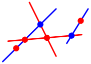
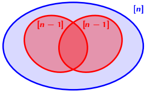


Proof in $C_{n-1}(n, q)$.



6 The Budapest effect

Before Budapest

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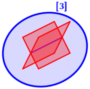

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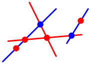

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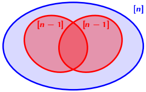

Bound: $\mathrm{wt}(c) \leqslant \mathbf{\frac{5}{2}} q^{n-1} - \frac{7}{5} q^{n-2}$

6 The Budapest effect

Before Budapest

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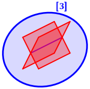

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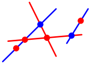

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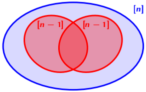

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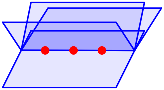
After Budapest

$(\geqslant \lfloor \sqrt{10q} \rfloor)$-secants.



6 The Budapest effect

Before Budapest

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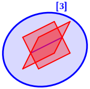

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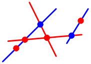

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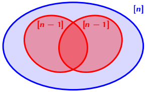

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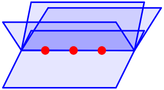

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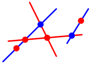

6 The Budapest effect

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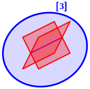

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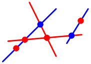

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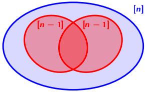

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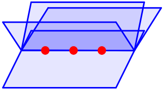

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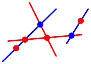

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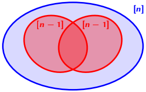
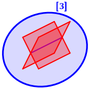


6 The Budapest effect

Before Budapest

Proof in $C_2(3, q)$.



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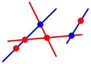

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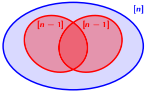

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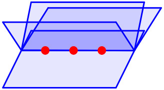

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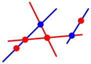

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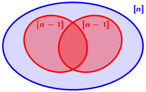

Bound: $\mathrm{wt}(c) < \mathbf{3} q^{n-1} - 3\theta_{n-2}$

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Before Budapest

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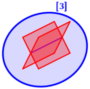

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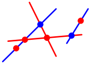

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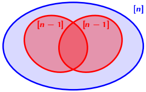

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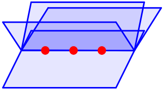

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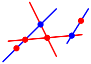

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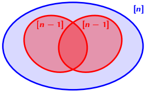

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Using *all* information, we can proof all lines are

$$(0, 1, 2, \mathbf{3} \,\|\, \boldsymbol{q-1}, q, q+1)\text{-secants,}$$

for all code words $c \in C_{n-1}(n, q)$,

$$\mathrm{wt}(c) \lesssim \boldsymbol{4q^{n-1} - \sqrt{10}q^{n-2}\sqrt{q}}.$$

7 The smallest weight code words of $C_2(3, p)$

To simplify things, we consider a code word $\boldsymbol{c} \in \boldsymbol{C_2(3, p)}$, with

$$2p^2 + p + 1 < \mathrm{wt}(c) \leqslant 4p^2 - \sqrt{10}p\sqrt{p} - \frac{31}{2}p - 21$$

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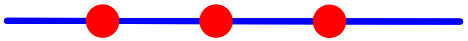
- There exists a 3-secant.



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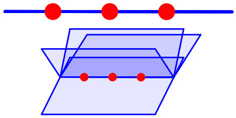
- There exists a 3-secant.
- The planes containing a 3-secant...



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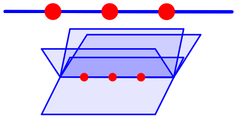

⋆ ... are all characterized.

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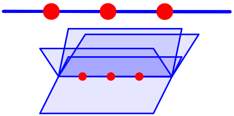
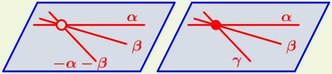


⋆ ... are all characterized.

⋆ ... are all of the *same* green type, ***or***...



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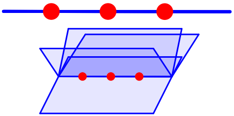
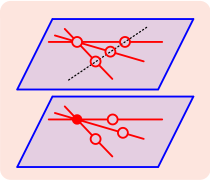
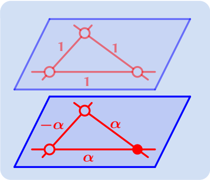


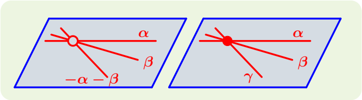


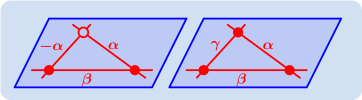
★ ... are all characterized.

★ ... are all of the *same* green type, ***or***...

★ ... can be divided into two types: a green type and another type.

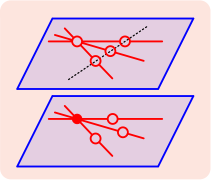


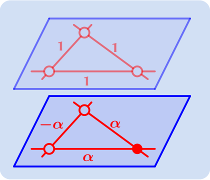


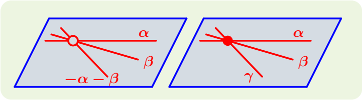


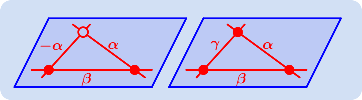
The smallest weight code words of $C_2(3, p)$

- ... are all characterized.
- ... are all of the *same* green type, ***or***...
- ... can be divided into two types:
  a green type and another type.









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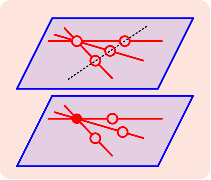

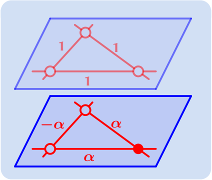

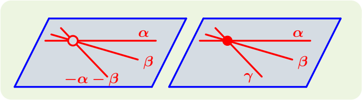

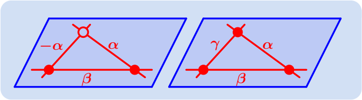

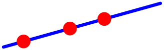

7 The smallest weight code words of $C_2(3, p)$

- ⋆ … are all characterized.
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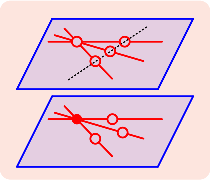

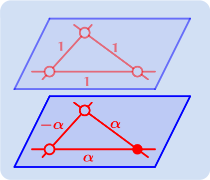

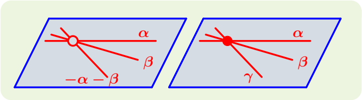

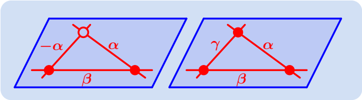

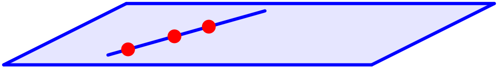

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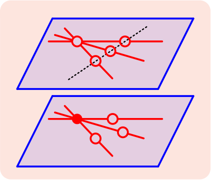

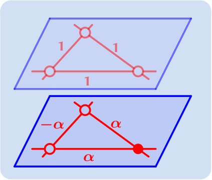

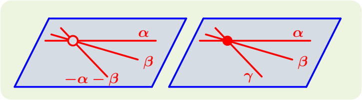

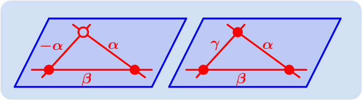

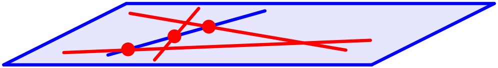

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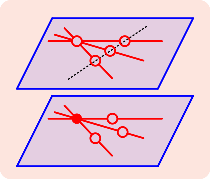

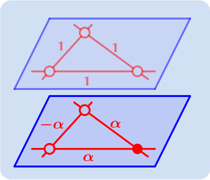

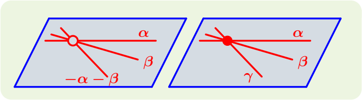

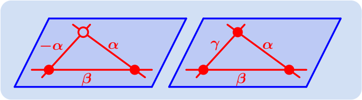

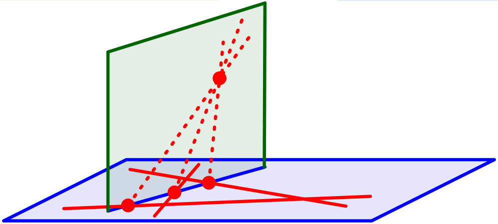

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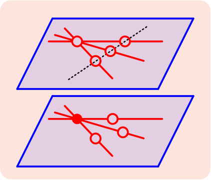

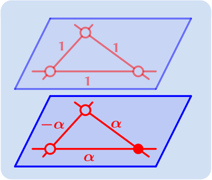

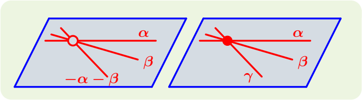

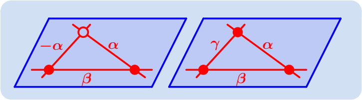

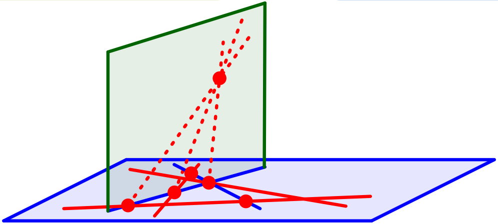

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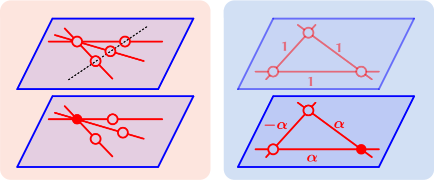

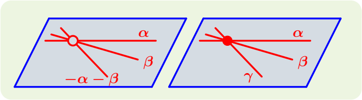

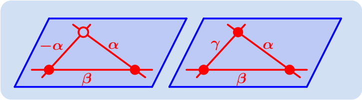

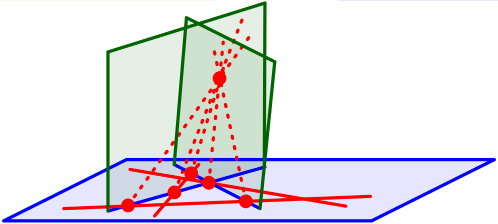

8 Results & further research

Our result: all small code words are cones

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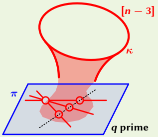

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Szőnyi & Weiner: the plane ($q = p^h$, $h \geqslant 2$, $q > 27$)

Code words of weight lower than $\frac{(p-1)(p-4)(p^2+1)}{2p-1}$, when $h = 2$,

$(\lfloor\sqrt{q}\rfloor + 1)(q + 1 - \lfloor\sqrt{q}\rfloor)$, when $h > 2$,

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Our result: further classification ($q = p^h$, $h \geqslant 2$, $q > 27$)

Code words up to weight $\left(\left\lfloor \frac{1}{2^{n-1}}\sqrt{q} \right\rfloor - \frac{9}{4}\right)\theta_{n-1}$, when $h = 2$,

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correspond to linear combinations of exactly $\left\lceil \frac{\mathrm{wt}(c)}{\theta_{n-1}} \right\rceil$ hyperplanes.

Fin.

Thank you for your attention. Are there any questions?

9 Questions & answers

Fin.ite geometry is awesome!

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