eSeminar UGent-VUB

Constructing saturating sets in projective spaces

using subgeometries

Lins Denaux

14th of January 2021

GHENT UNIVERSITY

Overview

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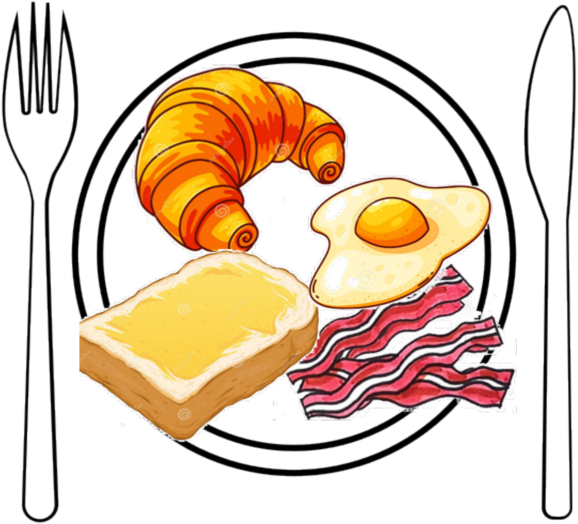
On the menu



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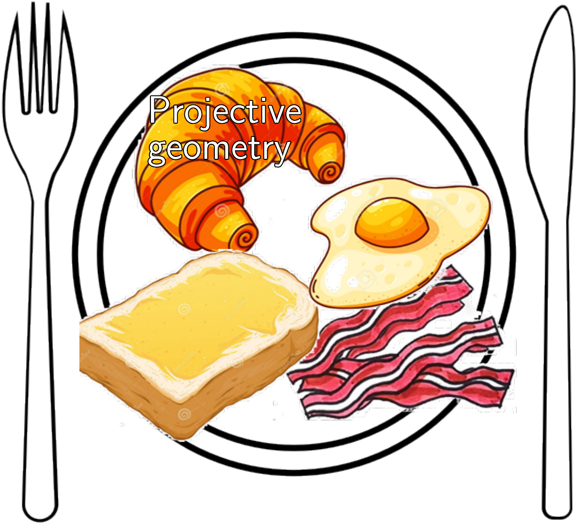

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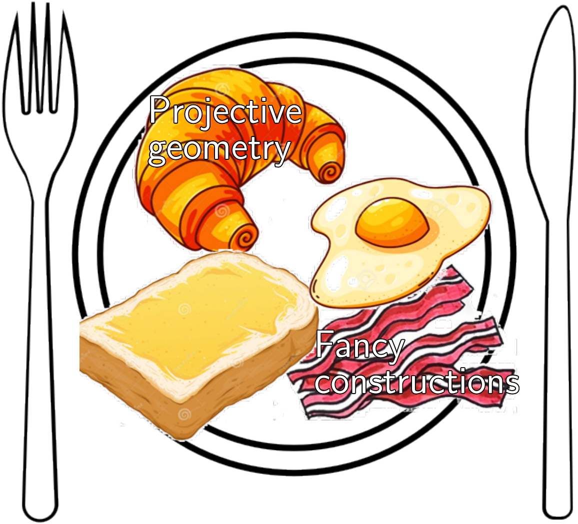

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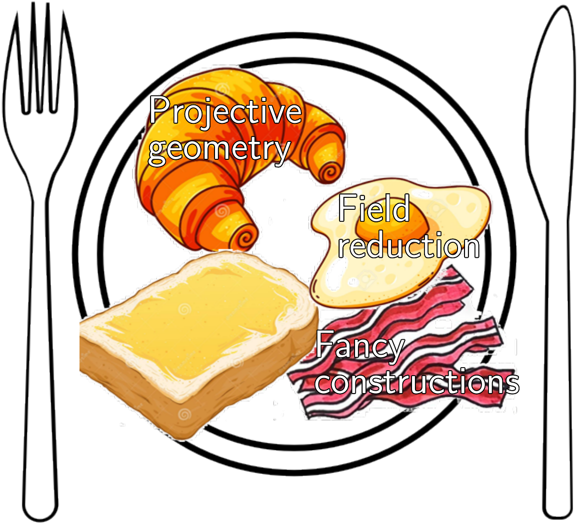
Projective geometry
Field reduction
Fancy constructions

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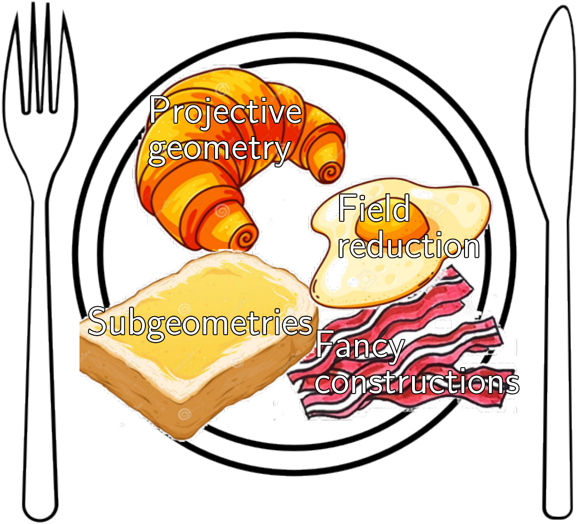
Projective
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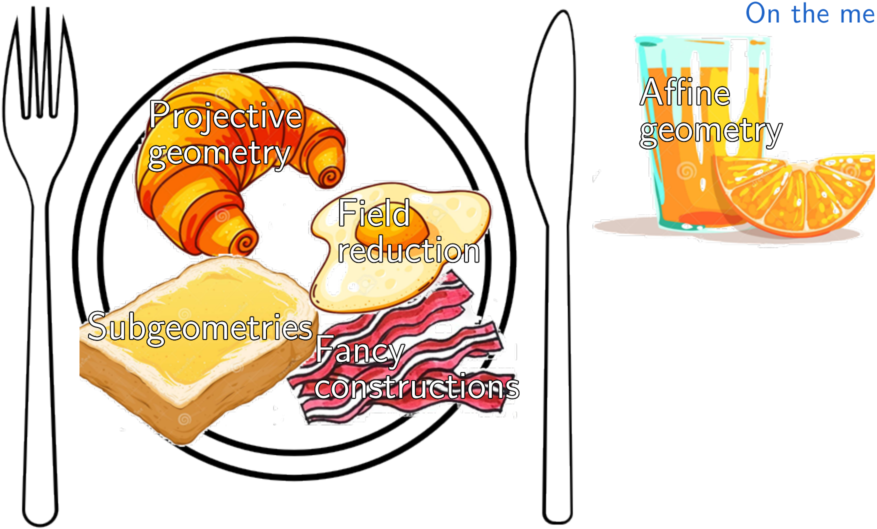
1 Introduction

On the menu



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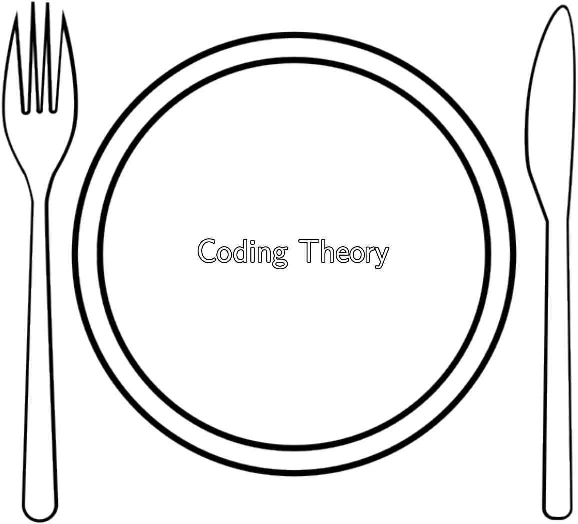

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1-saturating sets of PG(2, *q*)	

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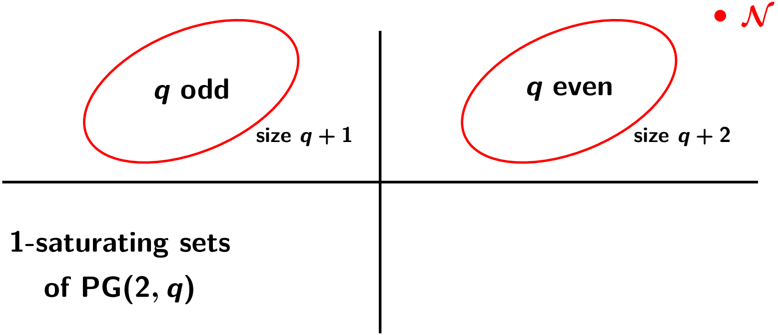

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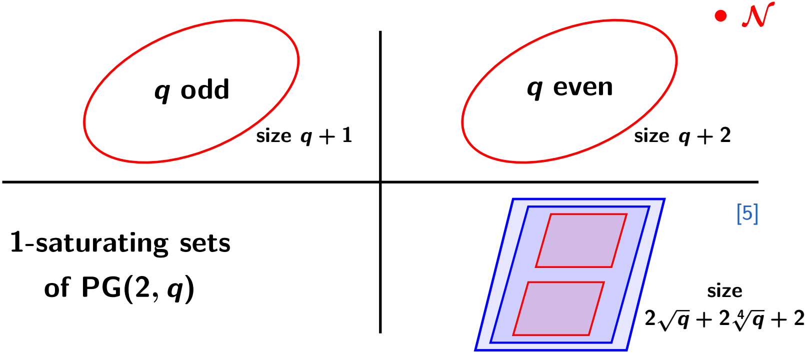

Definition and examples

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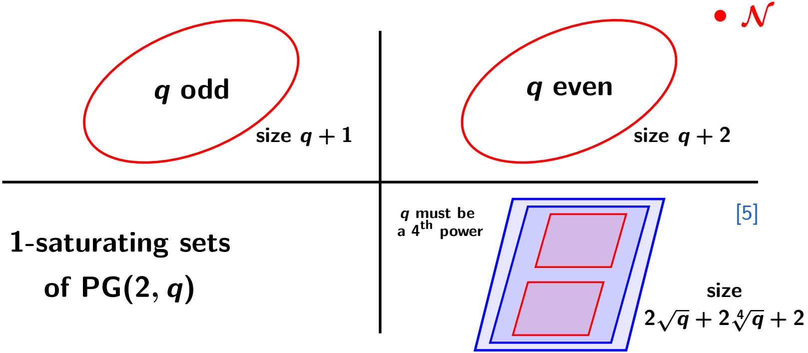

Saturating sets of $PG(3, q)$

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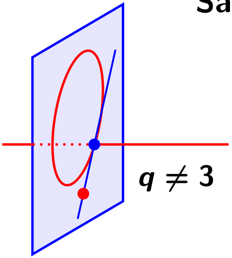

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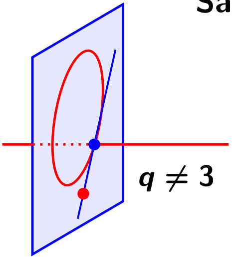

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- Size: $2q + 1$.

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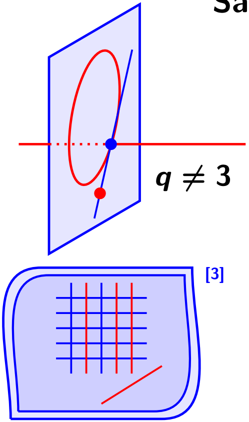

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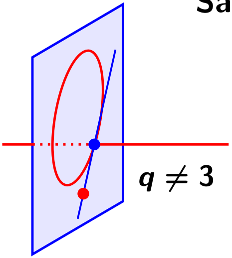

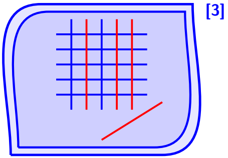

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- **2**-saturating set of PG$(3, q)$ [5].
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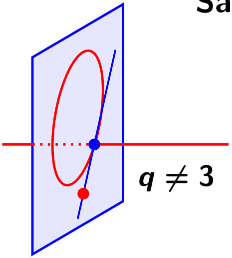

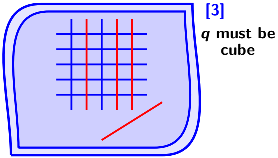

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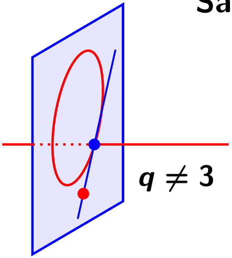

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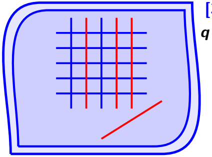

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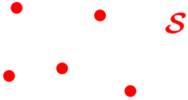
$\boldsymbol{s_q(n, \varrho)}$:= size of a smallest ϱ-saturating set of PG(n, q).

Correspondence with covering codes



ϱ-saturating set of $\mathrm{PG}(n,q)$, $\mathcal{S} := \{P_1, P_2, P_3, \ldots, P_{|\mathcal{S}|}\}$.

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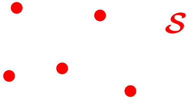

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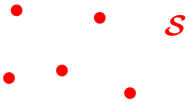

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coordinates of P_i

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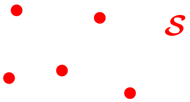

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PC-matrix of a $[|\mathcal{S}|, |\mathcal{S}| - n - 1]_q$ $(\varrho + 1)$-*covering code*!

Any vector of $\mathbb{F}_q^{|\mathcal{S}|}$ lies within Hamming distance $\varrho + 1$ of a codeword.

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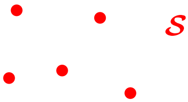

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Goal: Finding good upper bounds for $s_q(n, \varrho)$.

Proposition (LD, 2020 [9])
Let $\mathcal{S}$ be a ϱ-saturating set of PG(n, q). Then

$$|\mathcal{S}| > \frac{\varrho + 1}{e} \cdot q^{\frac{n-\varrho}{\varrho+1}} + \frac{\varrho}{2}.$$

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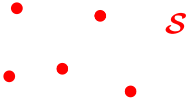

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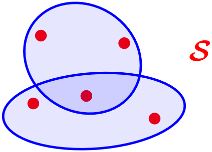

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$$\binom{|\mathcal{S}|}{\varrho+1}\theta_\varrho \geqslant \theta_n.$$

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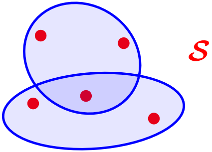

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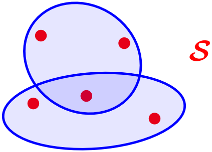

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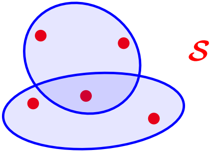

Keep in mind

$$s_q(n,\varrho) \quad \gtrsim \quad \boldsymbol{\varrho \cdot q^{\frac{n-\varrho}{\varrho+1}}}.$$

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$$\boldsymbol{s_q(n, \varrho) \lesssim \binom{n+1}{\varrho} q^{\frac{n-\varrho}{\varrho+1}}}$$

if q is a $(\varrho + 1)^{th}$ power.

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if n is even and n, q are large.

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Empty results from the past: a flashback

Theorem (Davydov & Östergård, 2000 [7])

$$s_q(n_1 + n_2 + 1, \varrho_1 + \varrho_2 + 1) \leqslant s_q(n_1, \varrho_1) + s_q(n_2, \varrho_2).$$

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Proof.

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Empty results from the past: a flashback

Theorem (Davydov & Östergård, 2000 [7])

$$s_q(n_1 + n_2 + 1, \varrho_1 + \varrho_2 + 1) \leqslant s_q(n_1, \varrho_1) + s_q(n_2, \varrho_2).$$

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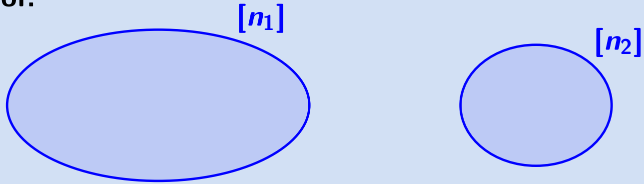

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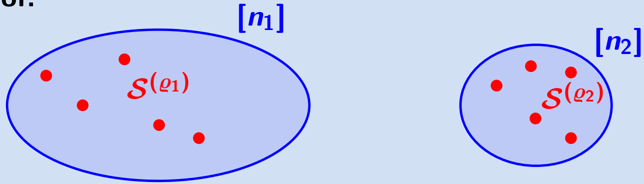

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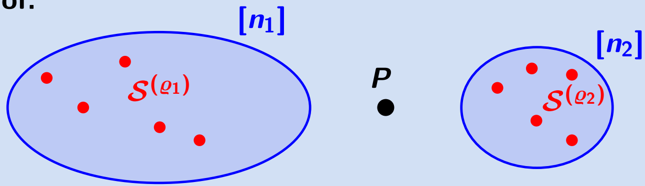

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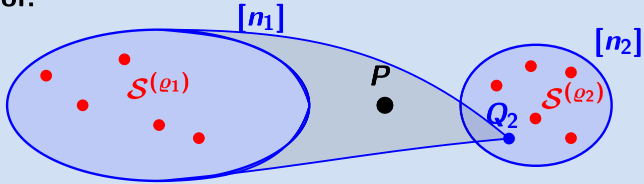

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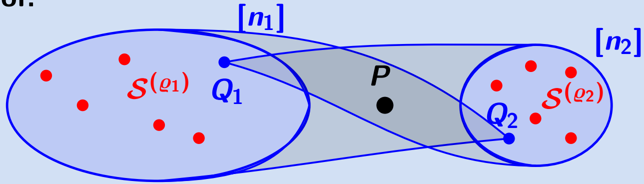

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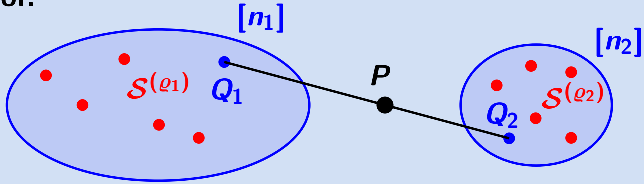

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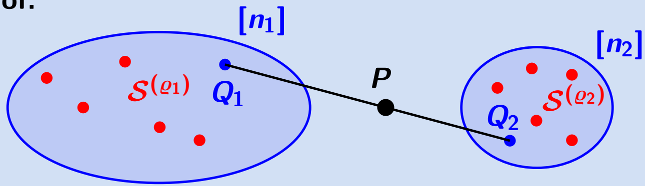

∎

Corollary

$$s_q(2\varrho + 1, \varrho) \leqslant (\varrho + 1)(q + 1).$$

Empty results from the past: a flashback

An inductive fiasco

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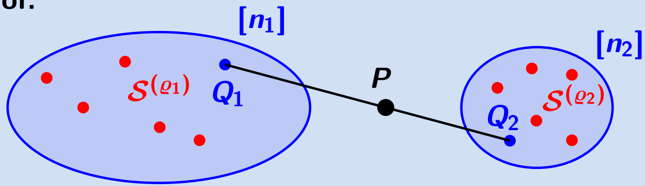

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Corollary

$$s_q(2\varrho+1,\varrho) \leqslant (\varrho+1)(q+1).$$

Keep in mind

$$s_q(n,\varrho) \quad \gtrsim \quad \varrho \cdot q^{\frac{n-\varrho}{\varrho+1}}.$$

Corollary

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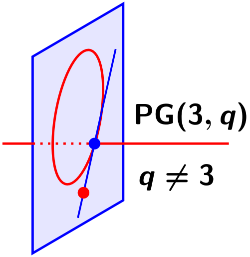

Theorem (Davydov [3])

$$s_q(3,1) \leqslant 2(q+1)-1.$$

Empty results from the past: a flashback

Corollary

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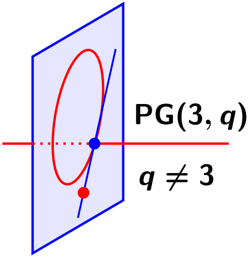

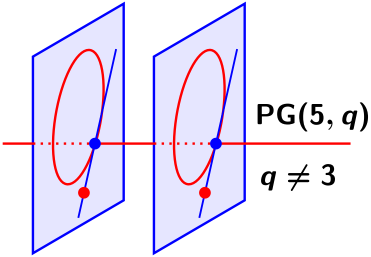

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Theorem (Davydov & Östergård, 2000 [7])

$$s_q(5, 2) \leqslant 3(q + 1) - 2.$$

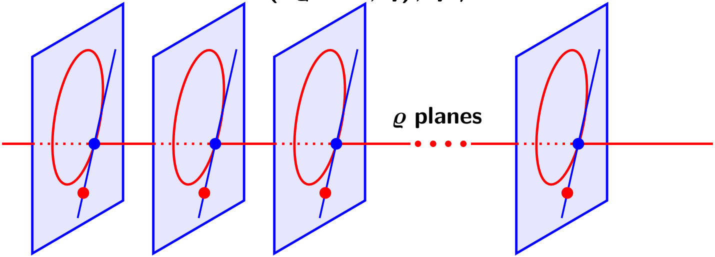
Corollary

$$s_q(2\varrho + 1, \varrho) \leqslant (\varrho + 1)(q + 1).$$

Keep in mind

$$s_q(n, \varrho) \quad \gtrsim \quad \boldsymbol{\varrho \cdot q^{\frac{n-\varrho}{\varrho+1}}}.$$

$\mathbf{PG(2\varrho + 1, q), q \neq 3}$



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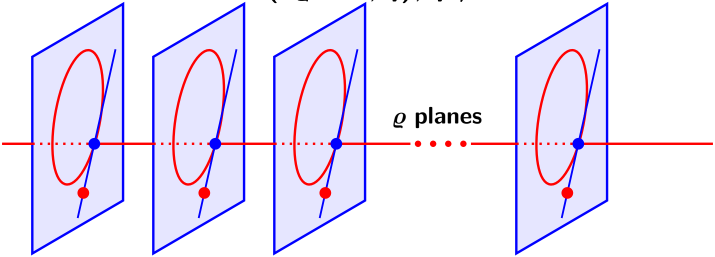

Theorem (LD, 2019)

If $q \neq 3$ and $\varrho \leqslant q$, then $s_q(2\varrho+1,\varrho) \leqslant (\varrho+1)(q+1) - \varrho$.

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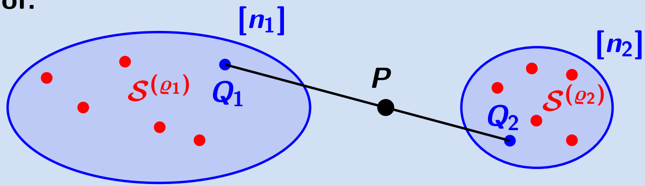

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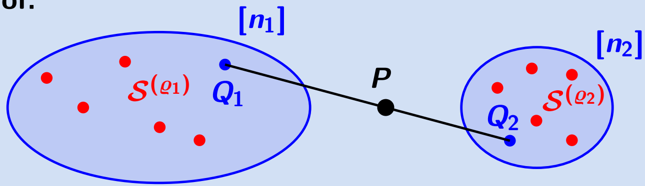

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$$s_q(2\varrho + 1, \varrho) \leqslant (\varrho + 1)(q + 1).$$

Corollary

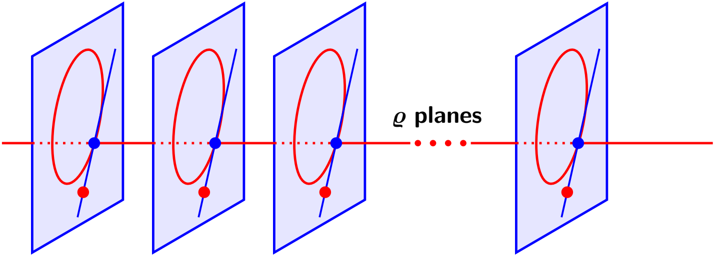
$$s_q\Big(k(\varrho + 1) + \varrho, \varrho\Big) \leqslant (\varrho + 1)\theta_k.$$

Empty results from the past: a flashback

Corollary

$s_q(2\varrho+1,\varrho) \leqslant (\varrho+1)(q+1)$.

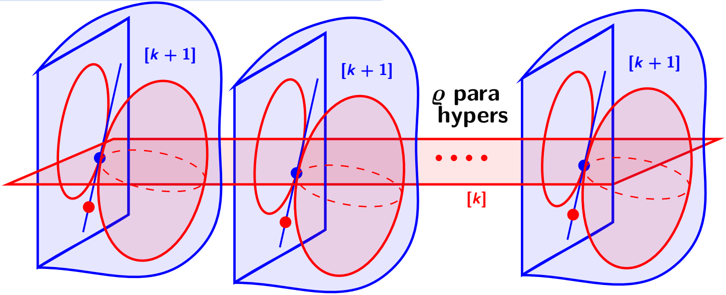
PG($2\varrho+1, q$)



Corollary

$s_q\Big(k(\varrho+1)+\varrho,\varrho\Big) \leqslant (\varrho+1)\theta_k.$

$\mathbf{PG}\Big(k(\varrho+1)+\varrho, q\Big)$



An inductive fiasco

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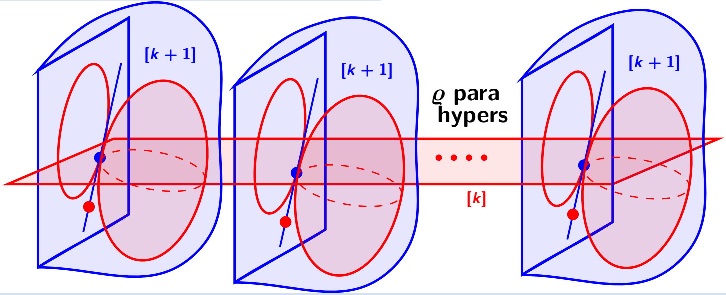

Theorem (LD, 2019)

Let $k \in \mathbb{N}$, $(k,q) \neq (1,3)$, $q \neq 2$ if k is even and $\varrho < \theta_k$. Then

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Keep in mind

$$s_q(n,\varrho) \quad \gtrsim \quad \boldsymbol{\varrho \cdot q^{\frac{n-\varrho}{\varrho+1}}}.$$

~~**Hypothesis (LD, 2019)**~~

Desperate wish (LD, 2020)

$$s_q(n,\varrho) \quad \lesssim \quad \boldsymbol{\varrho \cdot q^{\frac{n-\varrho}{\varrho+1}}},$$

for all n, $\varrho \leqslant n$ and ∞-many q.

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Empty results from the past: a flashback

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2 Empty results from the past: a flashback

An inductive fiasco

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... say what now?

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SpringerLink

Published: 10 June 2019

New covering codes of radius R, codimension tR and $tR+\frac{R}{2}$, and saturating sets in projective spaces

Alexander A. Davydov, Stefano Marcugini & Fernanda Pambianco

Designs, Codes and Cryptography **87**, 2771–2792(2019) | Cite this article

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Let $k \in \mathbb{N}$ be *large enough*, $q \geqslant 7$ and $\varrho \geqslant 3$. Then

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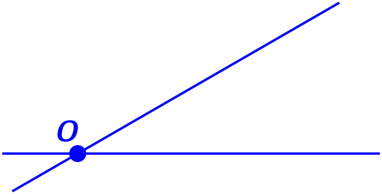

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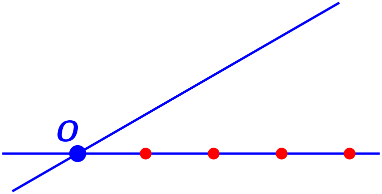

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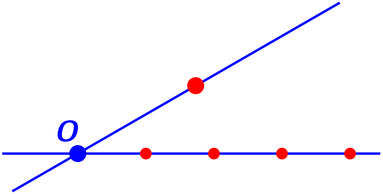

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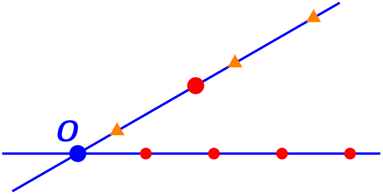

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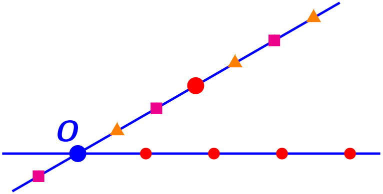

Empty results from the past: a flashback

A better but bitter start

- Let $n = 2$ and $\varrho = 1$.
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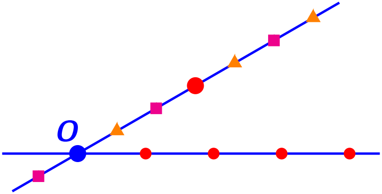

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$$s_q(2,1) \quad \gtrsim \quad \sqrt{\boldsymbol{q}}.$$

- 1-sat. set of $\mathrm{PG}(2,q)$.

Empty results from the past: a flashback

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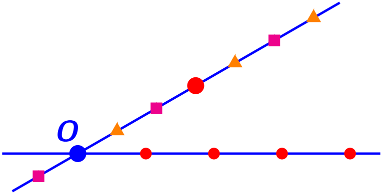

Theorem (Davydov, 1995 [3])
Let q be square. Then

$$s_q(2,1) \leqslant 3\sqrt{q} - 1.$$

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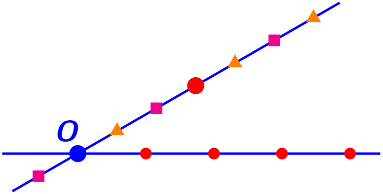

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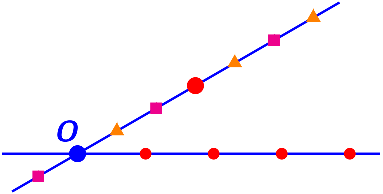

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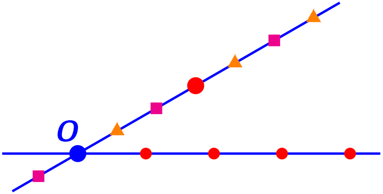

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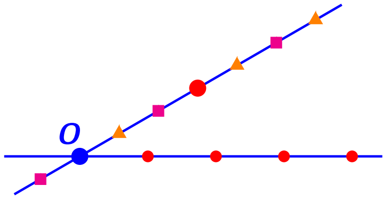
$$\sqrt{q}(\sqrt{q}+1) = q + q - \sqrt{q}(\sqrt{q}-1).$$

Empty results from the past: a flashback

A better but bitter start

- Let $n = 2$ and $\varrho = 1$.
- Let q be square.



Theorem (Davydov, 1995 [3])
Let q be square. Then

$$s_q(2,1) \leqslant 3\sqrt{q} - 1.$$

Keep in mind

$$s_q(2,1) \quad \gtrsim \quad \sqrt{\boldsymbol{q}}.$$

- 1-sat. set of PG$(2, q)$.
- Davydov: algebraic proof.
- Combinatorial proof?
  - $\sqrt{q}(\sqrt{q}+1)$ collinear Baer sublines through O.

$$\sqrt{q}(\sqrt{q}+1) = q+q-\sqrt{q}(\sqrt{q}-1).$$

Theorem (Davydov et al., 2011 [5])
Let q be a fourth power. Then

$$s_q(2,1) \leqslant 2\sqrt{q} + 2\sqrt[4]{q} + 2.$$

Empty results from the past: a flashback

- Let $n \in \mathbb{N}^\times$ and $\varrho = 1$.
- Let q be square.

Keep in mind

$$s_q(n,1) \quad \gtrsim \quad \boldsymbol{q}^{\frac{n-1}{2}}.$$

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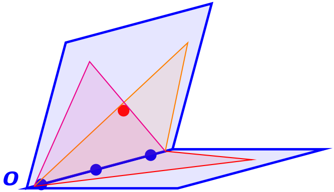

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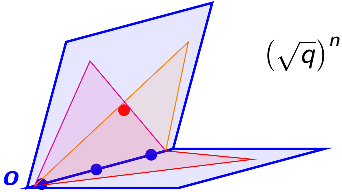

$$(\sqrt{q})^n(\sqrt{q}+1) = 2(\sqrt{q})^{n+1} - (\sqrt{q})^n(\sqrt{q}-1).$$

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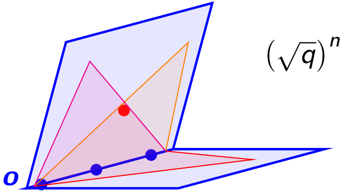

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Two extra obstacles:

1. What if P in 'plane'?
2. What if P in o?

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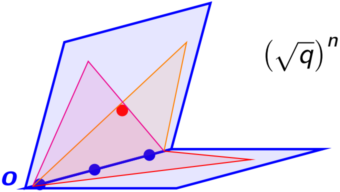

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Theorem (LD, 2019)
Let q be square. Then

$$s_q(n,1) \leqslant \begin{cases} 3q^{\frac{n-1}{2}} + 3q^{\frac{n-3}{2}} + \cdots + 3q^{\frac{1}{2}} - \frac{n}{2} & \text{if } n \text{ is even,} \\ 3q^{\frac{n-1}{2}} + 3q^{\frac{n-3}{2}} + \cdots + 3 - \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

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Let n be even and q be square. Then

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- Davydov et al., 2011 [5]

$$\boldsymbol{s_q(n,\varrho) \lesssim \binom{n+1}{\varrho} q^{\frac{n-\varrho}{\varrho+1}}}$$

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Constructions and families of nonbinary linear codes with covering radius 2

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A.A. Davydov **All Authors**

Published in: IEEE Transactions on Information Theory (Volume: 45 , Issue: 5, Jul 1999)

Page(s): 1679 - 1686

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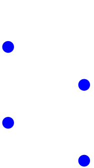

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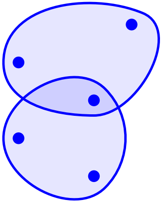

$[n-\varrho-1]$

- Choose $n+1$ independent points.
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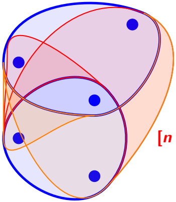

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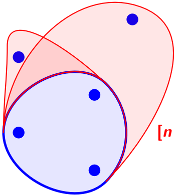

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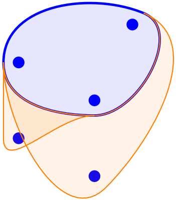

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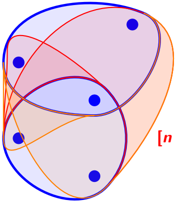

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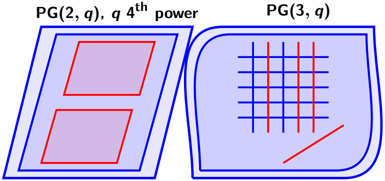

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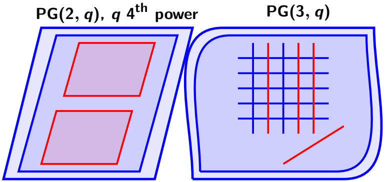

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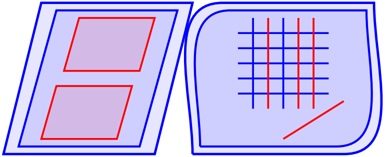
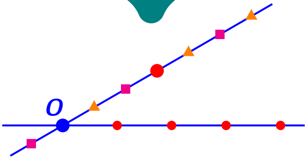
PG(2, q), q 4^{th} power **PG(3, q)**





3 The mixed subgeometry approach

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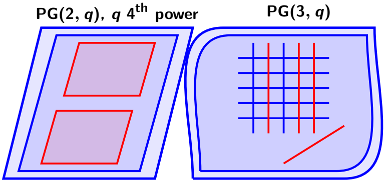

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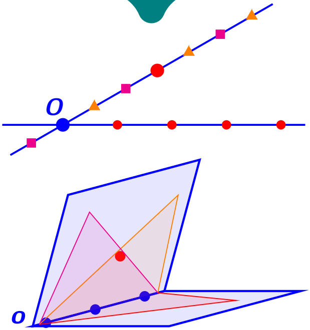

3 The mixed subgeometry approach

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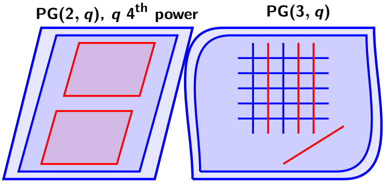

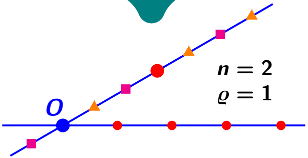

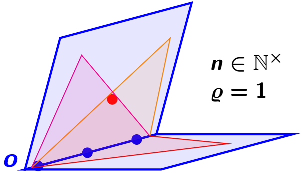

The mixed subgeometry approach

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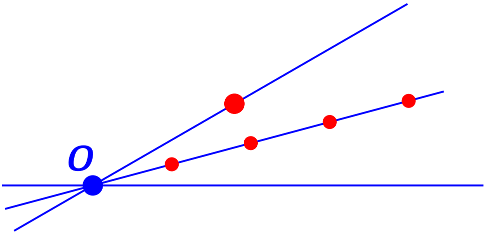

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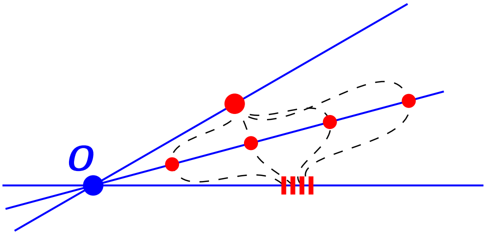

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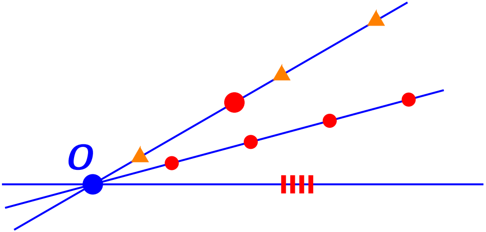

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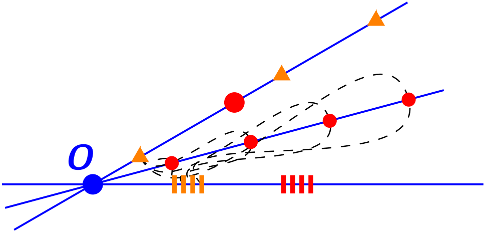

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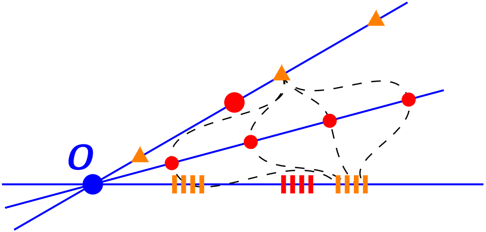

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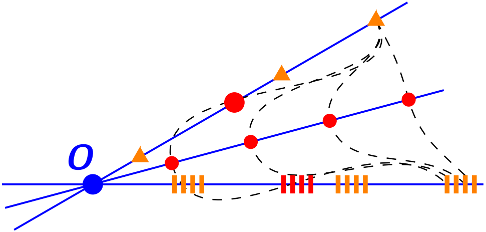

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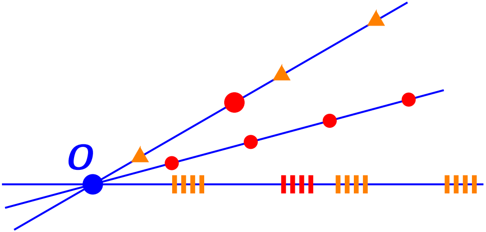

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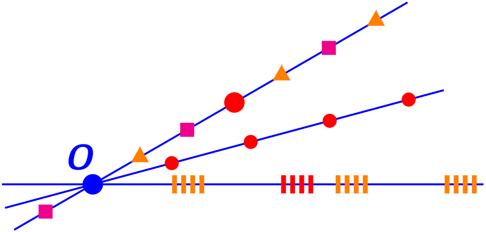

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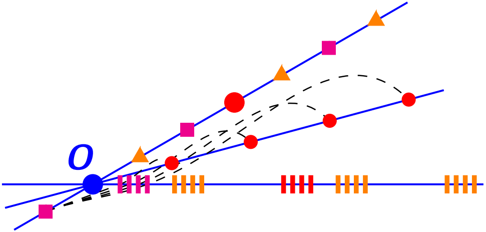

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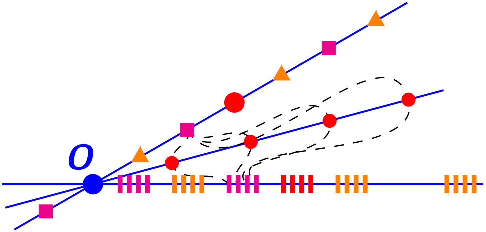

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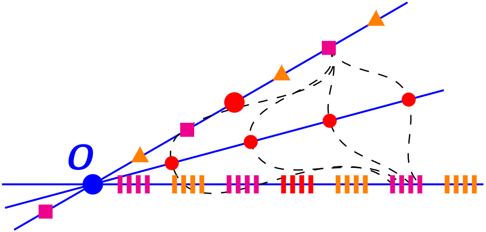

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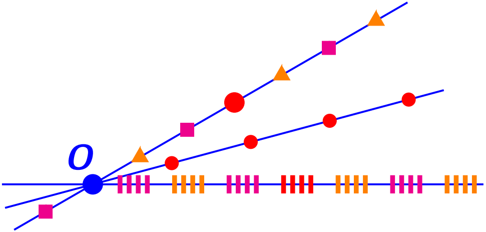

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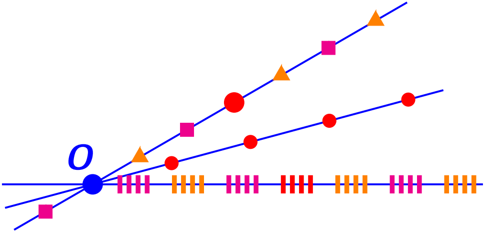

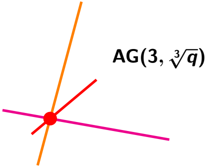

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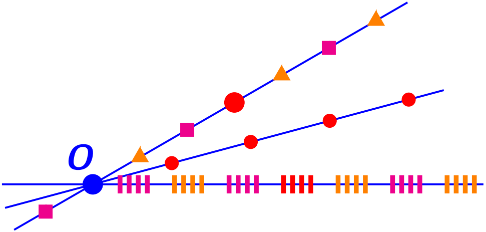

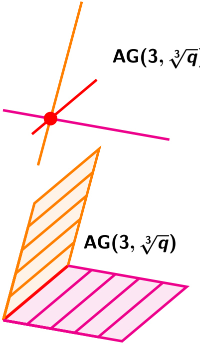

The mixed subgeometry approach

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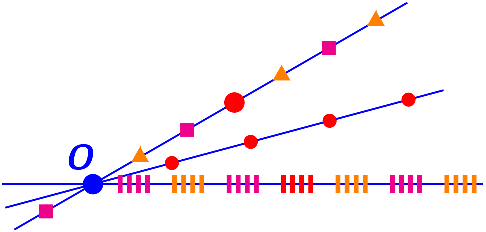

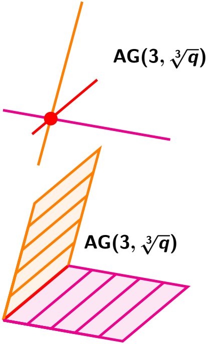

Theorem (LD, 2019)
Let q be cube. Then

$$s_q(3, 2) \leqslant 6\sqrt[3]{q} - 3.$$

3

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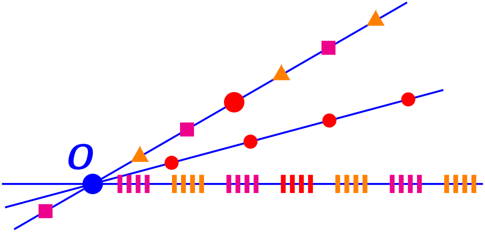

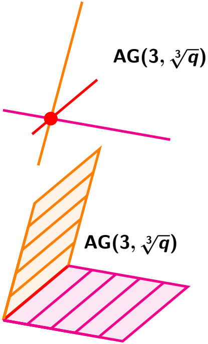

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Theorem (Davydov et al., 2011 [5])
Let q be cube. Then

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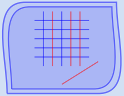

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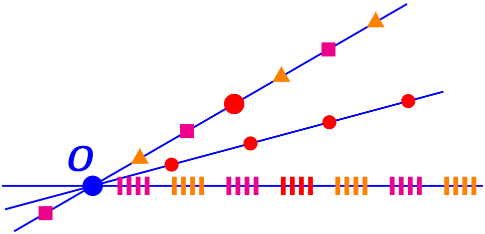

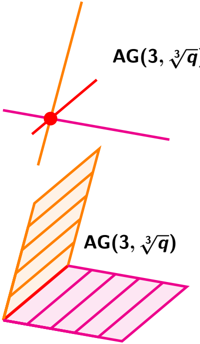

"This last construction looks promising!" - LD, 2019

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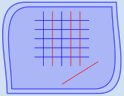

Subgeometries are affine lines

Suppose $q = (q')^{\varrho+1}$.

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Let $m \in \mathbb{N}^{\times}$, Σ hyperplane of $\mathrm{PG}(m, q)$. $\boldsymbol{Y(\varrho, m, q')} := (\mathcal{P}_{\text{sub}}, \mathcal{L}_{\text{sub}})$.

- $\mathcal{P}_{\text{sub}} := \mathrm{PG}(m, q) \setminus \Sigma$.
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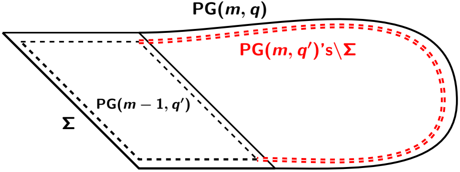

Subgeometries are affine lines

Two isomorphic point-line geometries

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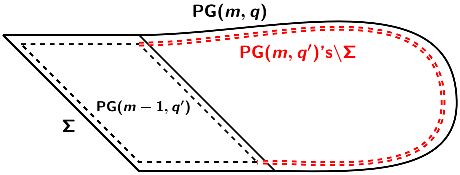

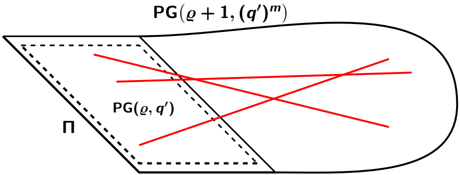

Let $m \in \mathbb{N}^{\times}$, Π hyperpl. of $\mathrm{PG}(\varrho+1, (q')^m)$. $\boldsymbol{T}^*(\mathbf{PG}(\boldsymbol{\varrho, q'})) := (\mathcal{P}_{\text{aff}}, \mathcal{L}_{\text{aff}})$.

- $\mathcal{P}_{\text{aff}} := \mathrm{PG}(\varrho+1, (q')^m) \setminus \Pi$.
- $\mathcal{L}_{\text{aff}} :=$ all affine parts of lines outside of Π intersecting $\mathrm{PG}(\varrho, q') \subseteq \Pi$.

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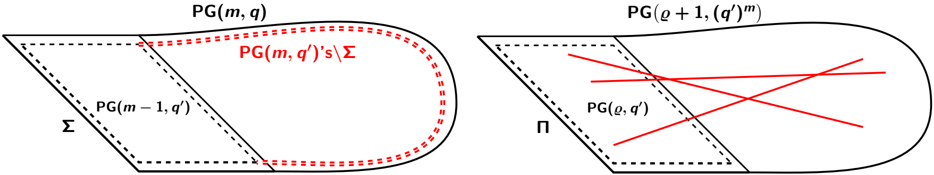

Theorem (LD, 2020 [9])

Subgeometries are affine lines, really.

Subgeometries are affine lines

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Theorem (LD, 2020 [9])

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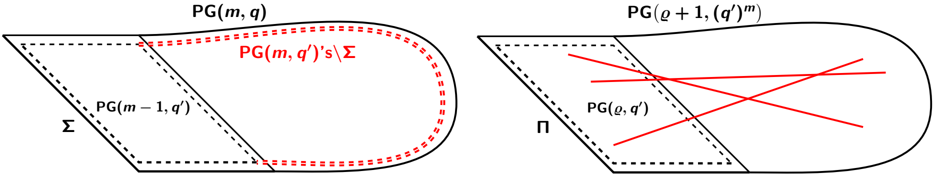

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Proof.

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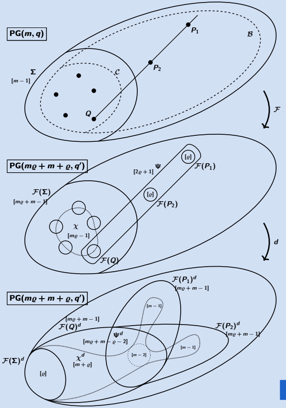

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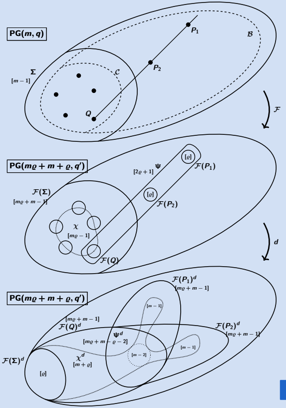

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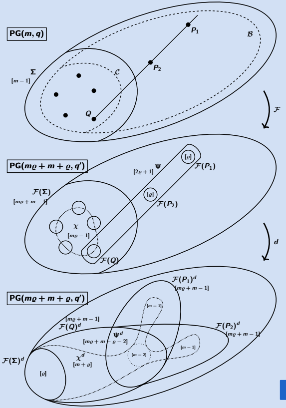
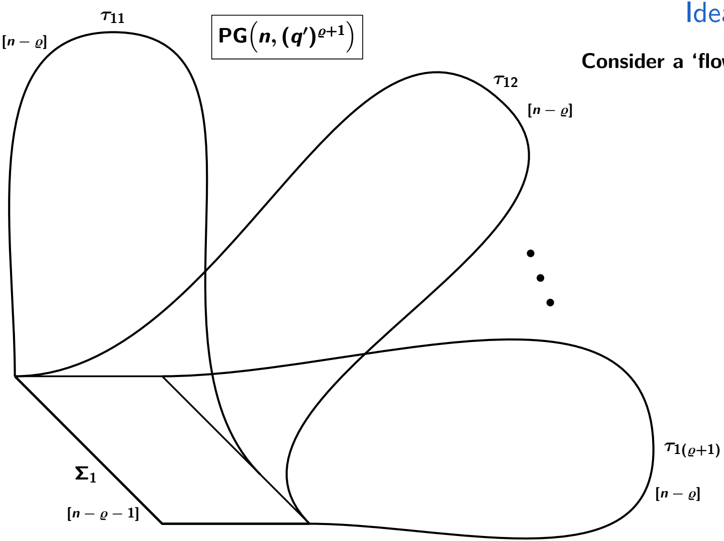


■

A construction for general n and ϱ

$\boxed{\mathbf{PG}\left(n, (q')^{\varrho+1}\right)}$

A construction for general n and ϱ



Consider a 'flower'.

A construction for general n and ϱ

Ideas and obstacles

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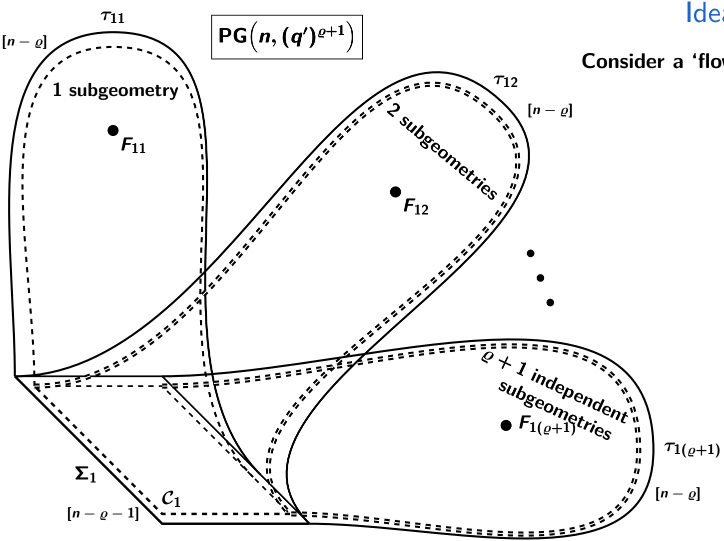
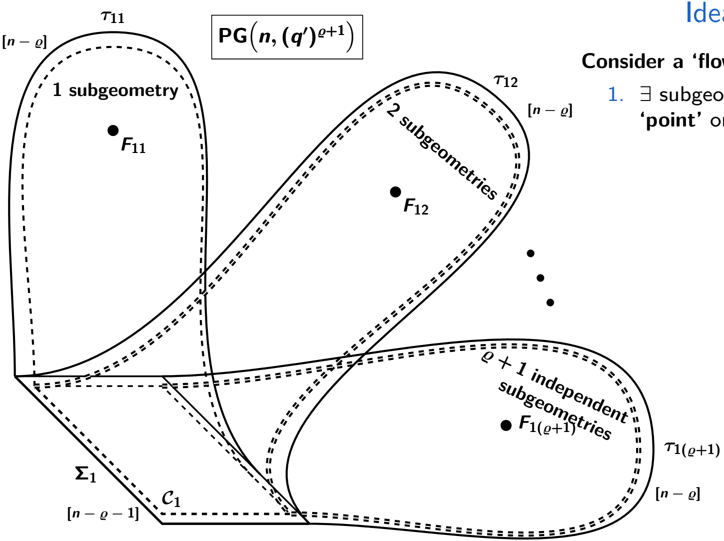


A construction for general n and ϱ



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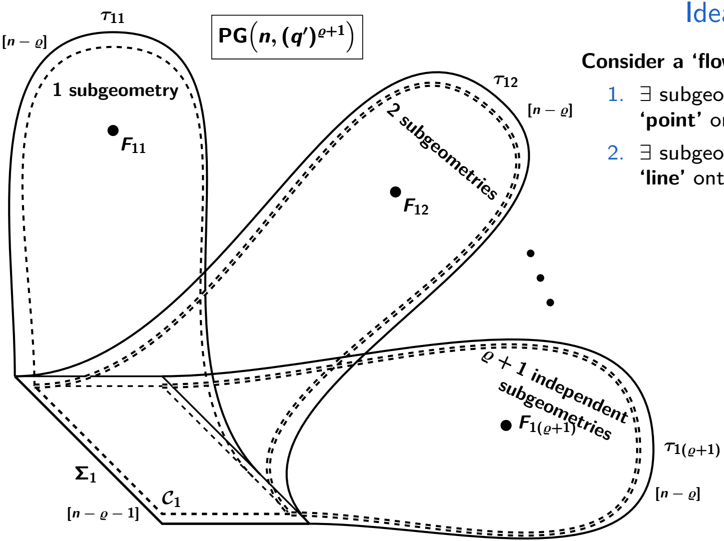
A construction for general n and ϱ



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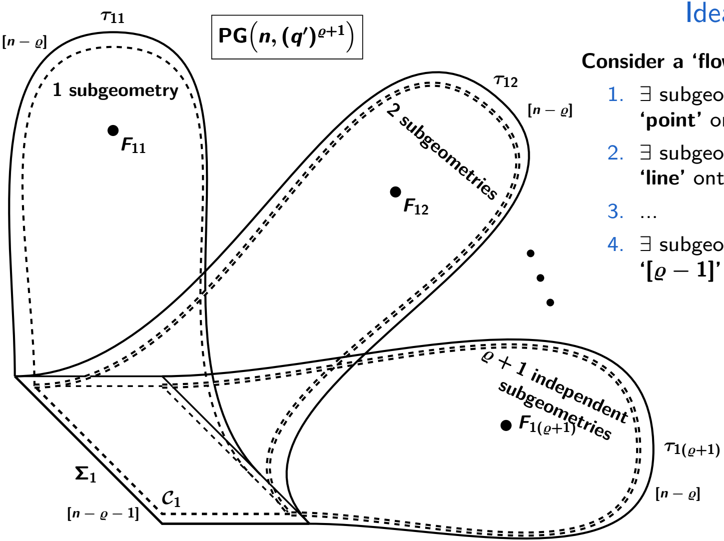
A construction for general n and ϱ

Ideas and obstacles



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A construction for general n and ϱ

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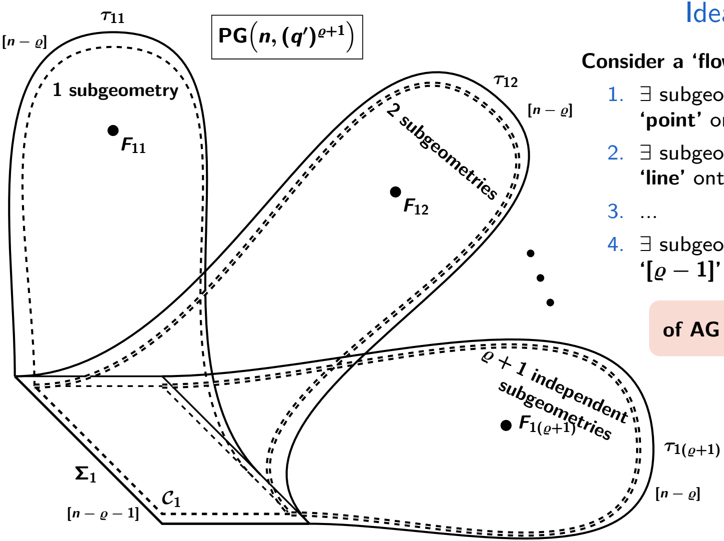

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A construction for general n and ϱ

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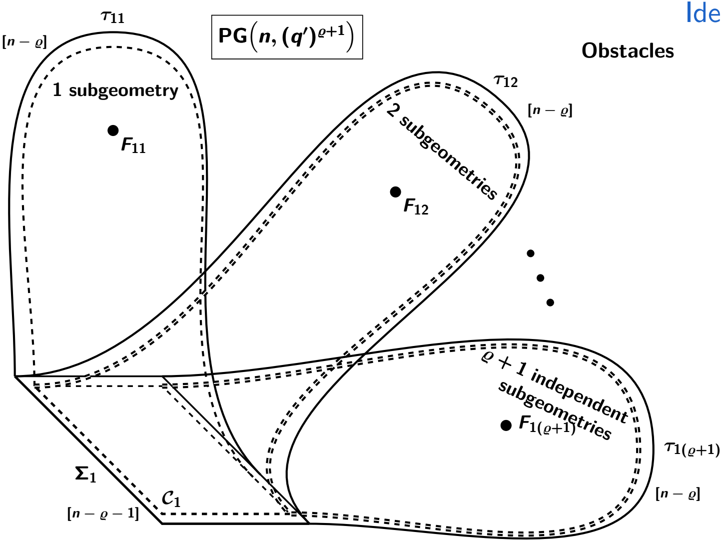
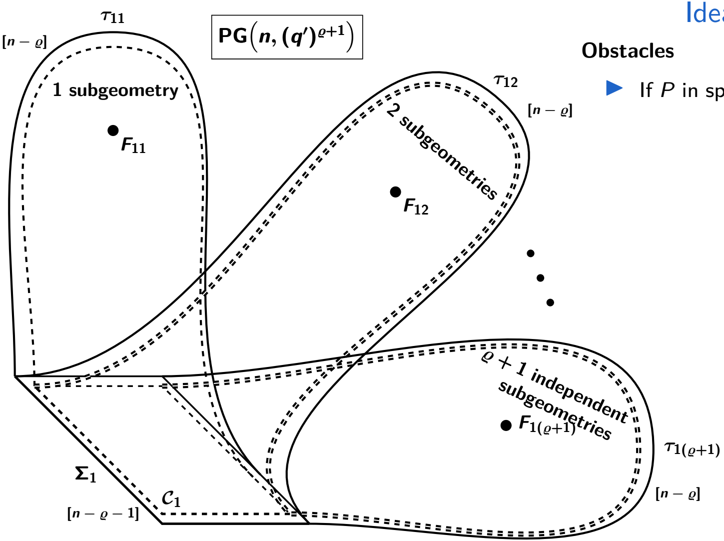


A construction for general n and ϱ



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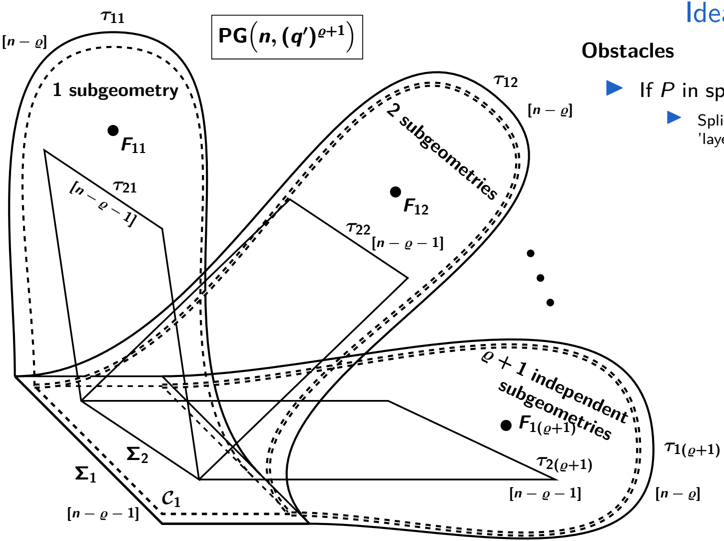
A construction for general n and ϱ



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A construction for general n and ϱ

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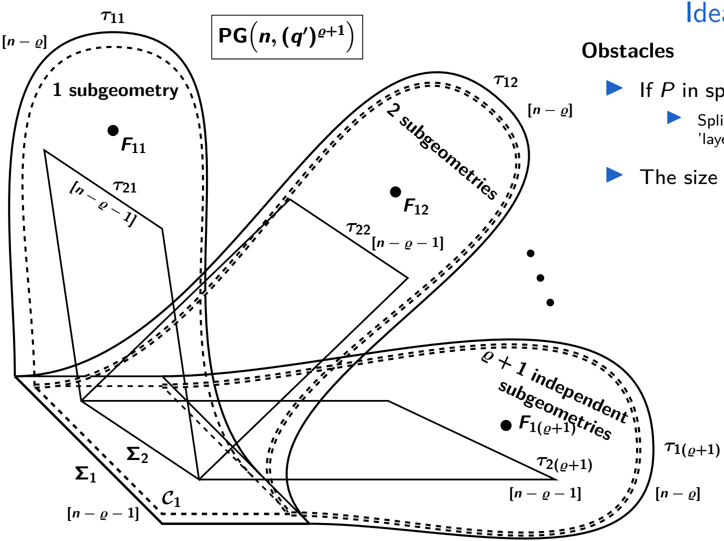

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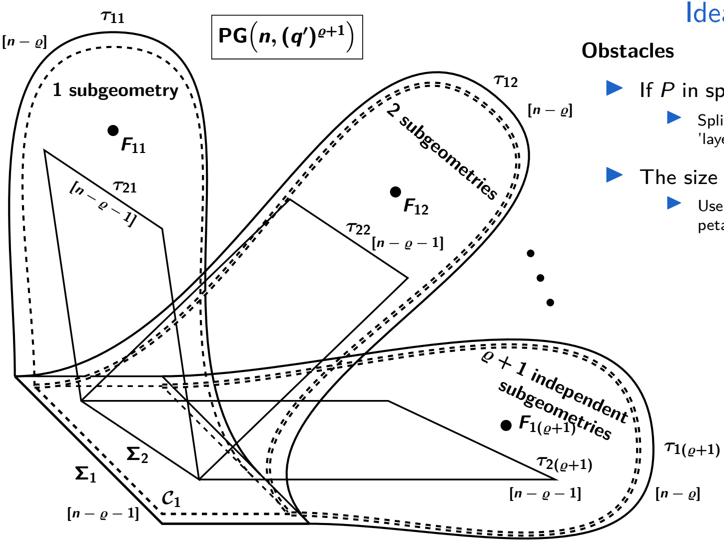

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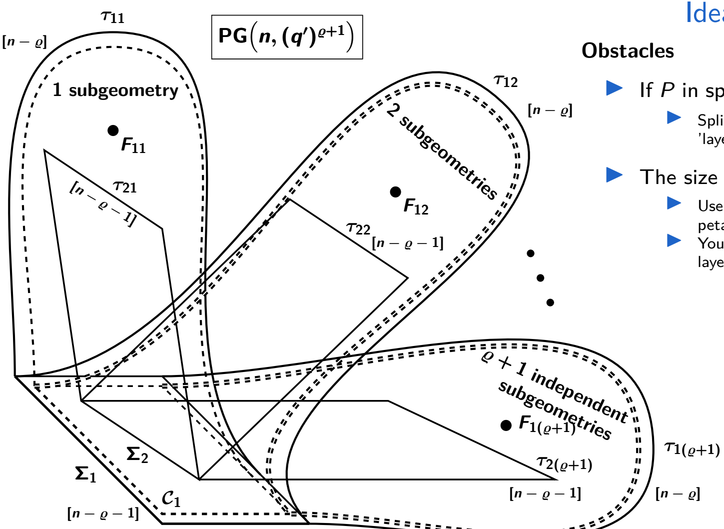

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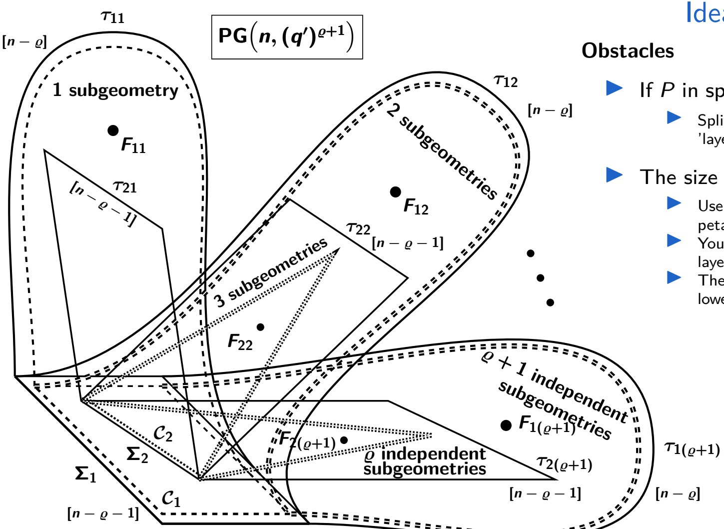

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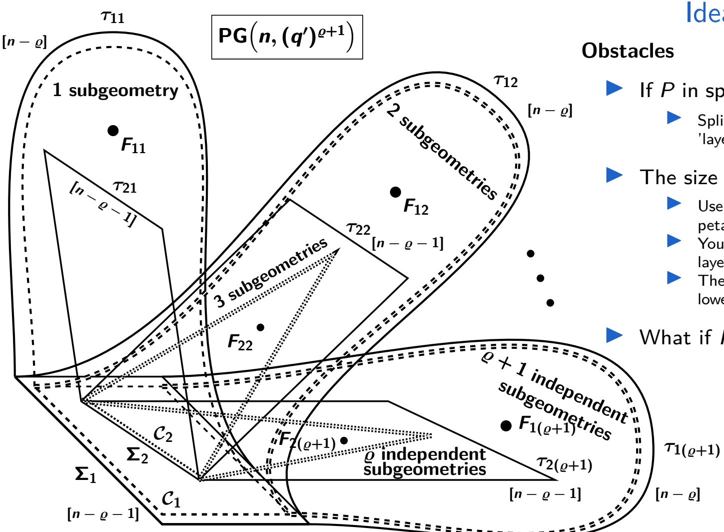

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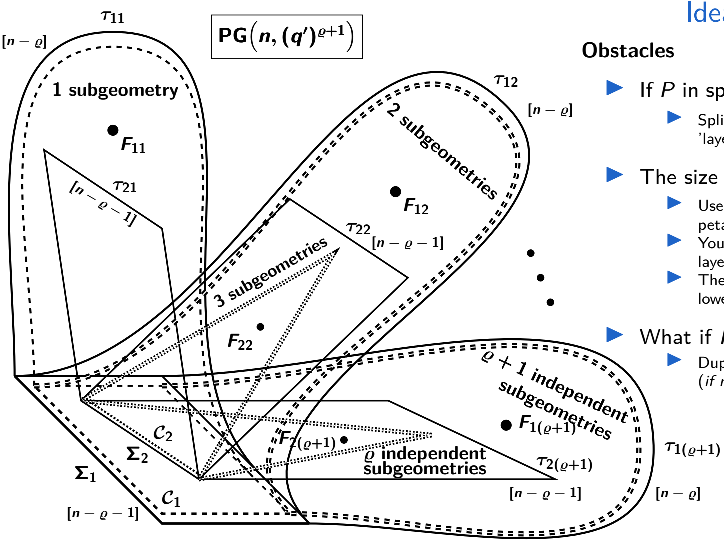

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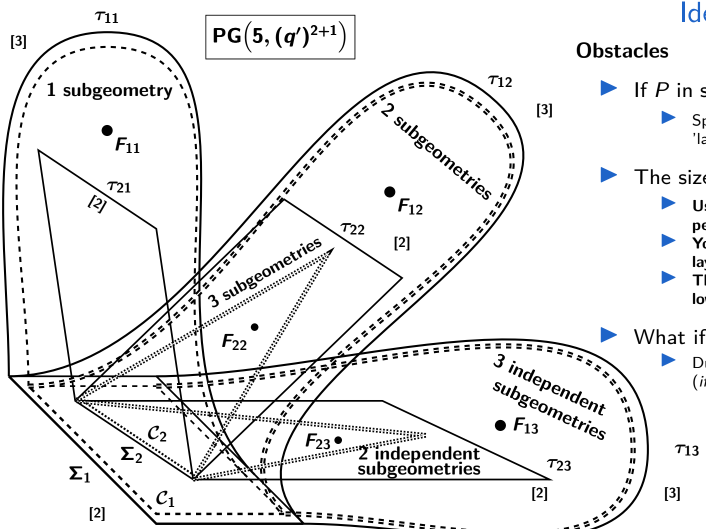

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  (*if necessary*)

A construction for general n and ϱ

Theorem (LD, 2020 [9])
Let $0 < \varrho < n$ and let $q = (q')^{\varrho+1}$ for any prime power q'. Then

$$s_q(n,\varrho) \leqslant \sum_{i=1}^{k(n,\varrho)} \left(\frac{(\varrho+1)(\varrho+2)}{2} (q')^{n+1-i(\varrho+1)} \right) + \sum_{i=1}^{k(n,\varrho)-1} \sum_{j=1}^{\varrho-1} \tilde{a}(\varrho,j)(q')^{n+1-i(\varrho+1)-j}$$
$$+ \sum_{j=1}^{\ell(n,\varrho)-1} \overline{a}(n,\varrho,j)(q')^{\ell(n,\varrho)-j} - \tilde{c}(n,\varrho) - \overline{c}(n,\varrho)$$
$$+ \delta_{q'=2} \cdot \left((2^{\varrho-1}-1) \cdot \sum_{i=1}^{k(n,\varrho)-1} \left(2^{n-\varrho+2-i(\varrho+1)} \right) + 2^{\ell(n,\varrho)} - 2 \right),$$

with

- $k(n,\varrho) := \left\lceil \frac{n-\varrho}{\varrho+1} \right\rceil$,
- $\ell(n,\varrho) := \left(n \pmod{\varrho+1} \right) + 1$,
- $\tilde{a}(\varrho,j) := \frac{\varrho(\varrho+2j+1)-j(3j+1)}{2}$,
- $\overline{a}(n,\varrho,j) := \frac{\ell(n,\varrho)\left(2\varrho-\ell(n,\varrho)+2j+1\right)-j(3j+1)}{2}$,
- $\tilde{c}(n,\varrho) := \left(k(n,\varrho)-1 \right) \frac{\varrho^2(\varrho+1)}{2}$,
- $\overline{c}(n,\varrho) := \frac{\varrho(\varrho+1)+\ell(n,\varrho)\left(\ell(n,\varrho)-1\right)\left(2\varrho-\ell(n,\varrho)+1\right)}{2}$,
- $\delta_{q'=2} := \begin{cases} 1 & \text{if } q'=2, \\ 0 & \text{if } q' \neq 2. \end{cases}$

A construction for general n and ϱ

Corollary (LD, 2020 [9])

Let $1 < \varrho < n$ and let $q = (q')^{\varrho+1}$ for any prime power q'. Then

$$s_q(n, \varrho) \leqslant \frac{(\varrho+1)(\varrho+2)}{2}(q')^{n-\varrho} + \varrho(\varrho+1)\left((q')^{n-\varrho-1} + \cdots + q' + 1\right).$$

A construction for general n and ϱ

One bound to rule them all

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Keep in mind

$$s_q(n, \varrho) \quad \gtrsim \quad \varrho \cdot q^{\frac{n-\varrho}{\varrho+1}}.$$

~~Hypothesis (LD, 2019)~~
Desperate wish (LD, 2020)

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for all n, $\varrho \leqslant n$ and ∞-many q.

A construction for general n and ϱ

One bound to rule them all

Corollary (LD, 2020 [9])
Let $1 < \varrho < n$ and let $q = (q')^{\varrho+1}$ for any prime power q'. Then

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if q is a $(\varrho+1)^{th}$ power.

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Coincidental example found in the wild:

Theorem (Davydov et al., 2011 [5])
Let $q = (q')^3$ for any prime power q'. Then

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Corollary (LD, 2020 [9])
Let $q = (q')^3$ for any prime power q'. Then

$$s_q(4, 2) \leqslant 6(q')^2 + 3q' - 6 \ (+2 \text{ if } q = 8).$$

A construction for general n and ϱ

arXiv:2008.13459

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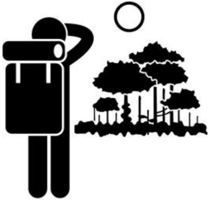

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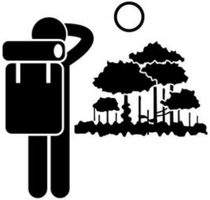

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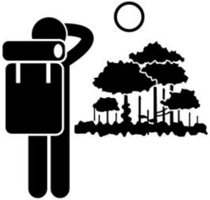

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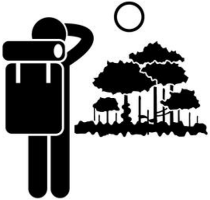

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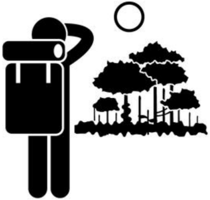

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A construction for general n and ϱ

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Slides: **https://users.ugent.be/~ldnaux**

Thank you for listening

Any questions?

Suggestions?

Funny anecdotes?

References

[1] **D. Bartoli, A. A. Davydov, M. Giulietti, S. Marcugini, and F. Pambianco.** *New bounds for linear codes of covering radius* 2, volume 10495 of *Lecture Notes in Comput. Sci.* Springer, Cham, 2017.

[2] **D. Bartoli, A. A. Davydov, M. Giulietti, S. Marcugini, and F. Pambianco.** *New bounds for linear codes of covering radii* 2 *and* 3, volume 11. 2019.

[3] **A. A. Davydov.** *Constructions and families of covering codes and saturated sets of points in projective geometry*, volume 41. 1995.

[4] **A. A. Davydov.** Constructions and families of nonbinary linear codes with covering radius 2. *IEEE Trans. Inform. Theory*, 45(5):1679–1686, 1999.

[5] **A. A. Davydov, M. Giulietti, S. Marcugini, and F. Pambianco.** *Linear nonbinary covering codes and saturating sets in projective spaces*, volume 5. 2011.

[6] **A. A. Davydov, S. Marcugini, and F. Pambianco.** New covering codes of radius R, codimension tR and $tR + \frac{R}{2}$, and saturating sets in projective spaces. *Des. Codes Cryptogr.*, 87(12):2771–2792, 2019.

[7] **A. A. Davydov and P. R. J. Östergård.** *On saturating sets in small projective geometries*, volume 21. 2000.

[8] **S. De Winter, S. Rottey, and G. Van de Voorde.** Linear representations of subgeometries. *Des. Codes Cryptogr.*, 77(1):203–215, 2015.

[9] **L. Denaux.** Constructing saturating sets in projective spaces using subgeometries. *arXiv:2008.13459*, 2020.