ON SOLUTIONS OF THE HIGHER SPIN DIRAC OPERATORS OF ORDER TWO

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Abstract. In this paper, we define twisted Rarita-Schwinger operators $\mathcal{R}^T_{l_1}$ and explain how these invariant differential operators can be used to determine polynomial null solutions of the higher spin Dirac operators $Q_{l_1,l_2}$. 
1 INTRODUCTION

Classical Clifford analysis is usually defined as a function theory generalizing complex analysis to the case of arbitrary dimension \( m \in \mathbb{N} \), where the role of the Cauchy–Riemann operator is played by an elliptic first order differential operator, see e.g. [4, 7, 9, 10]. One of these operators is the Dirac operator, being invariant with respect to the spin group. This operator is acting on spinor-valued functions, see e.g. [1, 6, 8].

Clifford analysis also offers an elegant framework to study function theoretical problems not only for the classical Dirac operator, but also for generalizations of it, acting on functions which take their values in arbitrary half-integer irreducible spin-representations, higher spin Dirac operators (HSD operators for short).

In a previous paper [11], we have already established their explicit definition. A special case is the operator \( Q_{l_1, l_2} \), studied in depth in e.g. [2, 3], which acts on polynomials taking values in irreducible \( \text{Spin}(m) \)-representations with highest weight \((l_1 + \frac{1}{2}, l_2 + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})\). We will introduce an alternative method to determine the polynomial null solutions for these operators, which can then be translated to the most general case.

The outline of this paper is as follows. In section 2, we will give some general Clifford analysis background in order to define the higher spin Dirac operators \( Q_{l_1, l_2} \) and the twisted Rarita-Schwinger operators \( R^T_{l_1} \). In section 3, we will then determine the structure of the polynomial null solutions of \( Q_{l_1, l_2} \).

2 CLIFFORD ANALYSIS BACKGROUND

Let \( \mathbb{R}_m \) be the Clifford algebra generated by an orthonormal basis \((\varepsilon_1, \ldots, \varepsilon_m)\) for the \( m \)-dimensional vector space \( \mathbb{R}^m \) and let \( \mathbb{C}_m = \mathbb{R}_m \otimes \mathbb{C} \) be its complexification. The multiplication is governed by the relations
\[
\varepsilon_i \varepsilon_j + \varepsilon_j \varepsilon_i = -2\delta_{ij}, \text{ for all } i, j = 1, \ldots, m
\]
If \( m = 2n + 1 \) is odd, we denote the unique spinor space, which can be realized as a minimal left ideal inside \( \mathbb{C}_m \), by \( \mathbb{S} \). In case \( m = 2n \), we have \( \mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^- \) with \( \mathbb{S}^\pm \) the space of positive (resp. negative) spinors. Unless explicitly stated otherwise, we will disregard the parity of the spinors in even dimension, and we will speak, with a slight abuse of language, about ‘the’ spinor space.

The Dirac operator, acting on \( \mathbb{S} \)-valued polynomials \( f(x) \), is defined as \( \partial_x = \sum_{j=1}^m x_j \partial x_j \).

For odd dimensions, the vector space \( \mathbb{S} \) defines the basic half-integer representation for the spin group \( \text{Spin}(m) \), described by the highest weight \((\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})\). In even dimensions, the spinor space \( \mathbb{S} \) is reducible, with the spaces \( \mathbb{S}^\pm \) both irreducible representations of \( \text{Spin}(m) \) with highest weights \( (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \) and \( (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}) \). It is crucial to mention that in the language of Clifford analysis, other irreducible half-integer \( \text{Spin}(m) \)-representations can be characterized as spaces of polynomials, see e.g. [5]. This is done using several vector variables \( u_i \in \mathbb{R}^m \). In particular, these irreducible modules will be modelled in terms of the following spaces, where we will denote the Dirac operators \( \partial_{u_i} \) by \( \partial_{\varepsilon} \).
These spaces will be characterized as particular classes of polynomials, which are introduced in the following definitions.

**Definition 1.** A function $f : \mathbb{R}^k \to \mathbb{S} : (u_1, \ldots, u_k) \mapsto f(u_1, \ldots, u_k)$ is called simplicial monogenic if it satisfies the system

$$\partial_i f = 0, \text{ for all } i = 1, \ldots, k$$
$$\langle u_i, \partial_j \rangle f = 0, \text{ for all } 1 \leq i < j \leq k$$

The vector space of $\mathbb{S}$-valued simplicial monogenic polynomials which are $l_i$-homogeneous in $u_i$ will be denoted by $\mathbb{S}_{l_1, \ldots, l_k}$, where we assume that $l_1 \geq \cdots \geq l_k$ (dominant weight condition). The following definition involves weaker conditions on the $\mathbb{S}$-valued functions, but will nevertheless be crucial in what follows.

**Definition 2.** A function $f : \mathbb{R}^k \to \mathbb{S} : (u_1, \ldots, u_k) \mapsto f(u_1, \ldots, u_k)$ is called monogenic if it satisfies $\partial_i f = 0$, for all $1 \leq i \leq k$.

The vector space of $\mathbb{S}$-valued monogenic polynomials which are $l_i$-homogeneous in $u_i$ will be denoted by $M_{l_1, \ldots, l_k}$, again with $l_1 \geq \cdots \geq l_k$. Each of these polynomial vector spaces can be seen as a module for the spin group, under the regular representation (or so-called $L$-representation) given by

$$L(s) P(x_1, \ldots, x_k) := s P(sx_1, \ldots, sx_k), \quad s \in \text{Spin}(m)$$

In e.g. [5], it was proven that under this action, the Spin($m$)-modules $\mathbb{S}_{l_1, \ldots, l_k}$ define a model for the irreducible highest weight representation characterized by means of

$$\mathbb{S}_{l_1, \ldots, l_k} \to \left( l_1 + \frac{1}{2}, \ldots, l_k + \frac{1}{2} \right) =: (l_1', \ldots, l_k').$$

Note that when $m = 2n$, one should also add a parity index to the spaces of simplicial monogenics, according to the one for spinors.

On functions taking values in the space $\mathbb{S}_\Lambda$, one can define operators playing the role of $\partial_x$ for $\mathbb{S}$-valued functions. These operators are precisely the HSD operators. These are first order Spin($m$)-invariant differential operators $Q_\Lambda$, where $\Lambda = (l_1, \ldots, l_k)$, which are uniquely defined up to a multiplicative constant:

$$Q_\Lambda : C^\infty(\mathbb{R}^m, \mathbb{S}_\Lambda) \to C^\infty(\mathbb{R}^m, \mathbb{S}_\Lambda) : f(x; u_1, \ldots, u_k) \mapsto Q_\Lambda f(x; u_1, \ldots, u_k)$$

The existence and uniqueness of these invariant differential operators follows from Fegan’s result [7]; their explicit form was determined in [11] as

$$Q_\Lambda = \left( \prod_{i=1}^k \left( 1 + \frac{u_i \partial_i}{m + 2l_i - 2i} \right) \right) \partial_x$$

Let us now define the twisted Rarita-Schwinger operator as follows:

**Definition 3.**

$$\mathcal{R}^T_{l_i} = 1 \otimes \mathcal{R}_{l_i} : C^\infty(\mathbb{R}^m, \mathcal{M}_{l_i} \otimes \mathcal{H}_{l_i}) \to C^\infty(\mathbb{R}^m, \mathcal{M}_{l_i} \otimes \mathcal{H}_{l_i})$$

where $\mathcal{H}_{l_i}$ is the space of harmonic polynomials in $u_i$, homogeneous of degree $l_i$. 

(1)
Remark 1. We call this operator twisted, as it is acting on spaces with the 'wrong' values.

In [11] the following result has been proven.

Proposition 1. For each \( f \in C^\infty(\mathbb{R}^m, S_{l_1,l_2}) \), it holds that
\[
R_{l_1}^T f = Q_{l_1,l_2} f + \pi_{l_1} [u_2] \langle \partial_2, \partial_x \rangle f \tag{2}
\]
where \( \pi_{l_1} [u_2] \) is an embedding operator, mapping \( S_{l_1,l_2-1} \)-valued polynomials to \( \mathcal{H}_{l_2} \otimes S_{l_1} \)-valued ones.

We will use this last result to determine the polynomial null solutions of the operator \( Q_{l_1,l_2} \) in the next section.

3 POLYNOMIAL SOLUTIONS IN THE CASE OF ORDER TWO

In view of the fact that \( S_{l_1,l_2} \subset \mathcal{M}_{l_1} \otimes \mathcal{H}_{l_2} \), the following holds:
\[
C^\infty(\mathbb{R}^m, S_{l_1,l_2}) \xrightarrow{R_{l_1}^T} C^\infty(\mathbb{R}^m, \mathcal{H}_{l_2} \otimes \mathcal{M}_{l_1})
\]
\[
C^\infty(\mathbb{R}^m, S_{l_1,l_2}) \xrightarrow{Q_{l_1,l_2}} C^\infty(\mathbb{R}^m, S_{l_1,l_2})
\]
\[
\langle \partial_2, \partial_x \rangle \xrightarrow{\mathcal{H}_{l_2} \otimes \mathcal{M}_{l_1}} C^\infty(\mathbb{R}^m, S_{l_1,l_2-1})
\]

We may then define two types of solutions. The type I solutions will be the solutions both in the kernel of \( R_{l_1}^T \) and \( \langle \partial_2, \partial_x \rangle \). The type II solutions are the remaining polynomial null solutions of \( Q_{l_1,l_2} \) which aren’t of type I.

Lemma 1. For any \( f \in C^\infty(\mathbb{R}^m, S_{l_1,l_2}) \) with \( R_{l_1}^T f = 0 \), one has that \( \langle \partial_2, \partial_x \rangle f = 0 \).

Proof. Suppose \( f \in C^\infty(\mathbb{R}^m, S_{l_1,l_2}) \) and \( R_{l_1}^T f = 0 \). As \( \langle u_2, \partial_2 \rangle f = 0 \), one also has that \( \partial_2 R_{l_1}^T f = 0 \). In view of the fact that
\[
\partial_2 R_{l_1}^T f = \left( -2 - \frac{4}{m+2l_1-2} \right) \langle \partial_2, \partial_x \rangle f
\]
where the constant is always different from 0, this proves the lemma.

From this lemma, it follows that the type I solutions of \( Q_{l_1,l_2} \) may be recharacterized as follows: \( f \in C^\infty(\mathbb{R}^m, S_{l_1,l_2}) \) is of type I if \( f \in R_{l_1}^T \).

Next, it is our goal to reveal more of the structure of the set of these type I solutions. Therefore, we need the following two lemmata. From now on, we will denote the degree of homogeneity in the variable \( x \) by \( h \). From now on, when acting on the function space \( C^\infty(\mathbb{R}^m, S_{l_1,l_2}) \), we will use the notation \( \cdot|_{S_{l_1,l_2}} \).

Lemma 2. For each \( f \) belonging to \( \ker R_{l_1}^T|_{S_{l_1,l_2}} \), one has that \( \langle \partial_1, \partial_x \rangle f \) belongs to \( \ker R_{l_1-1}^T|_{S_{l_1-1,l_2}} \).
Proof. Suppose that \( f \in \ker R^T_{l_1}|_{S_{l_1,l_2}} \). We can verify that \( \langle \partial_1, \partial_x \rangle f \in C^\infty(\mathbb{R}^m, S_{l_1,l_2}) \):
\[
\partial_1 \langle \partial_1, \partial_x \rangle f = \partial_2 \langle \partial_1, \partial_x \rangle f = 0
\]
since both sets of operators commute, and
\[
\langle u_1, \partial_x \rangle \langle \partial_1, \partial_x \rangle f = -\langle \partial_2, \partial_x \rangle f = 0
\]
due to Lemma 1. We then get that
\[
\mathcal{R}^T_{l_1-1} \langle \partial_1, \partial_x \rangle f = \left( 1 + \frac{u_1 \partial_1}{m + 2l_1 - 4} \right) \partial_x \langle \partial_1, \partial_x \rangle f = \langle \partial_1, \partial_x \rangle f = 0
\]
which proves the lemma.

\[\square\]

Lemma 3. For each \( f \) belonging to \( \ker h \mathcal{R}^T_{l_1}|_{S_{l_1,l_2}} \), one has that
\[
\langle \partial_1, \partial_x \rangle^{l_1-l_2+1} f = 0
\]

Proof. Suppose that \( f \in \ker h \mathcal{R}^T_{l_1}|_{S_{l_1,l_2}} \). Lemma 2 tells us that \( \langle \partial_1, \partial_x \rangle f \in \ker h^{-1} \mathcal{R}^T_{l_1-1}|_{S_{l_1-1,l_2}} \).

Recalling that \( l_1 \geq l_2 \), we thus get
\[
\langle \partial_1, \partial_x \rangle^{l_1-l_2} f \in \ker \mathcal{R}^T_{l_2}|_{S_{l_2,l_2}}.
\]

Since \( \langle \partial_2, \partial_x \rangle \langle \partial_1, \partial_x \rangle^{l_1-l_2} f = 0 \) in view of Lemma 1 we also have that
\[
0 = \langle u_2, \partial_1 \rangle \langle \partial_2, \partial_x \rangle \langle \partial_1, \partial_x \rangle^{l_1-l_2} f = (-\langle \partial_1, \partial_x \rangle + \langle \partial_2, \partial_x \rangle \langle u_2, \partial_1 \rangle) \langle \partial_1, \partial_x \rangle^{l_1-l_2} f = \langle \partial_1, \partial_x \rangle^{l_1-l_2+1} f
\]
which proves the lemma.

\[\square\]

From Lemma 2 we get that
\[
\ker h \mathcal{R}^T_{l_1}|_{S_{l_1,l_2}} \supseteq \left( \ker h \mathcal{R}^T_{l_1} \cap \ker h \langle \partial_1, \partial_x \rangle \right)|_{S_{l_1,l_2}} \oplus \langle \partial_1, \partial_x \rangle^{-1} \ker h^{-1} \mathcal{R}^T_{l_1}|_{S_{l_1-1,l_2}}
\]
where we use the notation \( \langle \partial_1, \partial_x \rangle^{-1} \) for the (not necessarily surjective) inverse operator of \( \langle \partial_1, \partial_x \rangle \). Note that
\[
\ker h \mathcal{Q}^T_{l_1} \cap \ker \langle \partial_1, \partial_x \rangle|_{S_{l_1,l_2}} = \{ f \in C^\infty(\mathbb{R}^m, S_{l_1,l_2}) : \partial_x f = 0 \} = \mathcal{M}_{h,l_1,l_2} \cap \ker \langle u_1, \partial_2 \rangle
\]
We will denote the latter space by \( \mathcal{M}^*_{h,l_1,l_2} \). Using Lemma 2 inductively, we get that
\[
\ker h \mathcal{R}^T_{l_1} \supseteq \bigoplus_{j=0}^{l_1-l_2} \left( \langle \partial_1, \partial_x \rangle^{-1} \right)^j \mathcal{M}^*_{h-j,l_1-j,l_2}
\]
where the sum stops at \( l_1 - l_2 \), because of Lemma 3.

Finally, we have that
Lemma 4. For each \( f \in C^\infty(\mathbb{R}^m, S_{1,l_2}) \), one has:

\[
f \in \ker Q_{l_1,l_2} \Rightarrow \langle \partial_2, \partial_x \rangle f \in \ker Q_{l_1,l_2-1}
\]

Proof. Suppose \( f \in C^\infty(\mathbb{R}^m, S_{1,l_2}) \) and \( f \in \ker Q_{l_1,l_2} \). Then, direct calculations yield that

\[
Q_{l_1,l_2-1} \langle \partial_2, \partial_x \rangle f = \frac{m + 2l_2 - 4}{m + 2l_2 - 6} \langle \partial_2, \partial_x \rangle Q_{l_1,l_2} f
\]

which proves the lemma.

Lemma 5. Suppose \( f \in C^\infty(\mathbb{R}^m, S_{1,l_2}) \) and \( f \in \ker Q_{l_1,l_2} \). One then has that

\[
f \in \ker \langle \partial_2, \partial_x \rangle^{j+1} \setminus \ker \langle \partial_2, \partial_x \rangle^j \Rightarrow \langle \partial_1, \partial_x \rangle^{l_1-l_2+1} \langle \partial_2, \partial_x \rangle^j f = 0
\]

Proof. Define the twistor operator \( T_{l_1,l_2}^{l_1-l_2} \) as follows:

\[
T_{l_1,l_2}^{l_1-l_2} := \langle \partial_1, \partial_x \rangle + \frac{\langle u_2, \partial_1 \rangle \langle \partial_2, \partial_x \rangle}{l_1 - l_2 + 2}
\]

It has been shown in e.g. [3] that \( T_{l_1,l_2}^{l_1-l_2} : C^\infty(\mathbb{R}^m, S_{1,l_2}) \rightarrow C^\infty(\mathbb{R}^m, S_{l_1-l_2,l_2}) \). We thus get, for each \( f \in C^\infty(\mathbb{R}^m, S_{1,l_2}) \cap \ker Q_{l_1,l_2} \), that

\[
T_{l_1,l_2} f := T_{l_2,l_2}^{l_2-l_1} \cdots T_{l_1-2,l_2}^{l_1-l_2} T_{l_1,l_2}^{l_1-l_2} f \in C^\infty(\mathbb{R}^m, S_{l_1,l_2})
\]

Therefore, we have that

\[
\langle u_2, \partial_1 \rangle T_{l_1,l_2} f = 0 \quad (3)
\]

It can be shown by means of direct calculations that for all \( g \in C^\infty(\mathbb{R}^m, S_{a,b}) \),

\[
\langle \partial_2, \partial_x \rangle T_{a,b}^{a-1,b} g = c T_{a,b-1}^{a-1,b-1} \langle \partial_2, \partial_x \rangle g
\]

where \( c = \frac{l_1 - l_2 + 2}{l_1 - l_2 + 1} \). Since \( \langle \partial_2, \partial_x \rangle^{j+1} f = 0 \), we then have that

\[
\langle \partial_2, \partial_x \rangle^j T_{l_1,l_2} f = C T_{l_1,l_2-j} \langle \partial_2, \partial_x \rangle^j f = C \langle \partial_1, \partial_x \rangle^{l_1-l_2} \langle \partial_2, \partial_x \rangle^j f \quad (4)
\]

where the constant \( C \) is not further explicited. On the other hand, we have that

\[
\langle \partial_2, \partial_x \rangle^{j+1} T_{l_1,l_2} f = 0
\]

whence also \( \langle u_2, \partial_1 \rangle \langle \partial_2, \partial_x \rangle^{j+1} T_{l_1,l_2} f = 0 \). Now,

\[
\langle u_2, \partial_1 \rangle \langle \partial_2, \partial_x \rangle^{j+1} T_{l_1,l_2} f = (-j+1) \langle \partial_1, \partial_x \rangle \langle \partial_2, \partial_x \rangle^j + \langle \partial_1, \partial_x \rangle^{j+1} \langle u_2, \partial_1 \rangle T_{l_1,l_2} f
\]

Due to (3) and (4), we thus have that

\[
\langle \partial_1, \partial_x \rangle^{l_1-l_2+1} \langle \partial_2, \partial_x \rangle^j f = 0
\]

which completes the proof.
We define the following grading on $\ker_h Q_{l_1,l_2}$. If we set

$$\ker_h^a Q_{l_1,l_2} := \ker_h(Q_{l_1,l_2}, \langle \partial_2, \partial_x \rangle^a) \setminus \ker_h(Q_{l_1,l_2}, \langle \partial_2, \partial_x \rangle^{a-1})$$

we get that

$$\ker_h Q_{l_1,l_2} = \bigoplus_{j=0}^{l_2} \ker_h^j Q_{l_1,l_2}$$

On account of Lemma 5, we then obtain, omitting the embedding factors,

$$\ker_h Q_{l_1,l_2} \supseteq \bigoplus_{i=0}^{l_1-l_2} \bigoplus_{j=0}^{l_2} M_{h_{-i-j},l_1-i,l_2-j}^i$$

Dimensional analysis done in [3] then reveals that the above inclusion in fact is an equality.

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